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SPECTRAL THEORY OF SEMIBOUNDED OPERATORS AND THEIR USE IN SPECTRAL ANALYSIS OF DIFFERENTIAL OPERATORS

(Parts I and II)

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SPECTRAL THEORY OF SEMIBOUNDED OPERATORS AND THEIR USE IN SPECTRAL ANALYSIS OF DIFFERENTIAL OPERATORS *[/465]

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The direct methods of the calculus of variations are applied to solve the eigenvalue problems of linear partial differential equations that have no conventional discrete spectrum, and to thus obtain the spectral theory of quantum-theoretical energy operators, based on Schrödinger's representation. The notations of the general operator theory of the "abstract" Hilbert space are used as basis for developing the spectral theory of semibounded symmetric operators. Hilbert's and Weyl's criteria for proving that the spectrum is partly discrete are extrapolated to semibounded operators. The theory is applied to differential operators, for the typical case of n = 1, 2, 3. It is demonstrated that the eigenelements of the projection operators are twice continuously differentiable functions. A method is given for an accurate determination of the nature of the spectrum, for the auxiliary potential v.

The present investigation was induced by the desire to use the direct methods of the calculus of variations for solving the eigenvalue problems of such linear partial differential equations that have no ordinary discrete spectrum and thus are not accessible to the calculus of variations. Specifically, we meant to obtain the spectral theory of the quantum-theoretical energy

* Numbers in the margin indicate pagination in the original foreign text.

operators, based on Schrödinger's representation.

In this attempt, it was found that a large number of equivalent conclusions and concepts in the various problems can be uniformly combined by subjecting them to the symbolism of the general operator theory of the Hilbert space; specifically, we mean here the "abstract" Hilbert space as it had originally been logically worked out by v.Neumann (Bibl.7.2). The previously preferred representation of the Hilbert space by infinitely many variables was found too awkward for the representation of function spaces. In addition, as mentioned earlier by v.Neumann (Bibl.7.1), the representation of unbounded linear operators by infinite matrices may actually be misleading.

Conversely, it was found unnecessary to make use of the general spectral theory of unbounded operators as developed by v.Neumann since we only took semibounded operators into consideration; for this type, the spectral theory can be directly reduced to that of the bounded operators. In fact, most energy operators are semibounded downward. Similarly, in treating eigenvalue differential equations by the calculus of variations, the semiboundedness was utilized to its major extent.

A theory for semibounded operators was developed by A.Wintner (Bibl.13); however, this refers mainly to infinite matrices.

Until now, eigenvalue problems of differential equations had frequently been reduced to the Hilbert theory [see for example (Bibl.6)] but in such a manner that the explicitly known Green function was used for finding the bounded reciprocal of the differential operator. Usually, cases were involved in which a discrete spectrum occurred. /466

Beyond this, the theory of differential equations with singularities, as developed by H.Weyl, (Bibl.12.1, 12.2)* was predominantly applied.

* (next page)

Weyl's theory was formulated in a more general manner by Stone (Bibl.10.2) and expanded differently, without reduction to integral operators.

Essentially, this theory uses the well-known two-parameter family of solutions of differential equations; this is the reason for the difficulty encountered in attempting a direct extrapolation to partial differential equations**.

In Part I of this report, we are developing the spectral theory of semi-bounded symmetric operators in the abstract Hilbert space. This theory is readily obtained when considering not only the operators but also the corresponding forms. Without restriction, let this form G be positive-semibounded, i.e., let it be assumed that a positive γ exists so that, with the unitary form H, the following is valid:

$$G \geqq \gamma H.$$

The form G can be conceived as the dimensional form of a new Hilbert space

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* Weyl treated the eigenvalue equation

$$0 = n\gamma - nb + \left(\frac{xp}{np} d\right) \frac{xp}{p} -$$

for a function u(x) in $x \geq 0$. Here, it is assumed that $p > 0$ and that p and q are continuous in $x \geq 0$. At $x = 0$, a boundary condition $\cos \vartheta\, u + \sin \vartheta \frac{du}{dx} = 0$ is established. Weyl demonstrated that two cases can occur:

1) the critical point case in which, in addition to the existence of $\int_0^\infty u^2 dx$, no further condition need be made for u at $x = \infty$;
2) the limit cycle case in which a cyclic one-parameter family of boundary conditions at $x = \infty$ is available for selection.

Weyl also gave a method for obtaining the eigenfunctions of the continuous spectrum from the solutions of the eigenvalue differential equation, not located in the Hilbert space, by integration to the eigenvalue.

** In writing this paper, we noted that Carleman (Bibl.1) mentioned that it is easy to apply the Weyl theory to equations with several variables, either directly or over the theory of the Hermitian integral equations.

(a subspace of the original space). Then H becomes a bounded form. The spectral analysis, known for the bounded form, thus leads directly to the spectral analysis of the semibounded* form G. /467

Such semibounded forms G can always be obtained from semibounded operators. The spectral analysis of such operators, however, is possible and can be obtained in this manner if and only if these operators are "selfadjoint" (hypermaximal). This condition, for the case of semibounded operators, can be replaced by considerably weaker types which are also easier to verify for our differential operators.

Finally, we will extrapolate Hilbert's and Weyl's criteria (Bibl.5.2, 12.3) for partial discreteness of the spectrum, to semibounded operators.

In Part II of our paper, the above theory will be applied to differential operators. There, we are restricting ourselves to two typical cases. Let the operator be

$$-\sum_{r=1}^{n} \frac{\partial^2}{\partial x_r^2} + v(x_1, \ldots, x_n),$$

applicable to the functions $f(x_1, \ldots, x_n)$. Only the case $n = 1, 2, 3$ will be continued to the end. First, we treat the case (1) of the infinite region with the steady function v, bounded downward; secondly, in this case (2) a singularity of v in one point is admitted; thirdly, a finite region is used along whose boundary the function f [case (3)] or its normal derivative is to vanish [case (4)]. For convenience, this particular region is selected as line segment, circle, and sphere. The results for the cases of the finite region are not new,

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* This also offers a simple access to the spectral theory of arbitrary unbounded selfadjoint linear operators. This new method, compared to the conventional methods by v.Neumann (Bibl.7.2), Stone (Bibl.10.1), and Fr.Riesz (Bibl.9), offers the main advantage that it does not presuppose that the basic Hilbert space be complex; this method will be used elsewhere.

but their treatment was included here so as to demonstrate in how far the theory of all cases can be developed in common.

The first problem is to indicate the spaces of the permissible functions. At first, these are no Hilbert spaces but are continued to Hilbert spaces by an adjunction of ideal elements. We will not realize these ideal elements by quadratically integrable functions according to Lebesgue, mainly for the reason that it can be demonstrated that the "eigenelements" of specific interest /468 here belong already to the initial function spaces*.

Similarly, the operator is first explained only in a space of twice differentiable functions and then is closed off formally but uniquely.

The main problem is to demonstrate that this operator is selfadjoint. In fact, this constitutes the essential difficulties of the entire theory; these can be overcome by extrapolating the method of reasoning developed specifically by Courant (Bibl.2.1, 2.3, 2.6, 4.1), which is decisive in the direct methods of the calculus of variations.

It will be found that, by coordination with the abstract operator theory, not much can be saved in the theories required for concrete differential operators. Aside from the somewhat more systematic arrangement, one gains the possibility of simultaneously treating cases in which a discontinuous spectrum occurs.

The main result of Part II is the spectral analysis of the differential operator. By this we mean the existence of a "spectral family", namely, a family of projection operators in the sense of the general spectral theory (see Part I, Sect.4). In addition, it is demonstrated that the eigenelements of

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* This process corresponds fully to the method used by Hilbert in his reduction of the integral equations to equations with infinitely many variables (Bibl.5.1).

these projection operators are twice continuously differentiable functions. We will not bother to represent these projection operators and their eigenelements by means of solutions of the eigenvalue differential equations. However, for the eigenfunctions of the point eigenvalues, it follows directly that they satisfy the eigenvalue differential equation. According to Weyl, the eigenfunctions of the continuous spectrum in the case of one dimension could be obtained from the solutions of the eigenvalue differential equation by integrating to the eigenvalue. In the separable cases with more dimensions, this is also entirely possible, as we will show in another paper. For such a representation, in the general case of higher dimensionality, no arguments are available; we also do not believe that this particular point need be emphasized in the investigation.

Another result relates to a discussion of the spectrum. Under simple conditions for the "accessory potential" v, the nature of the spectrum can be more accurately defined. We have here a discrete point spectrum growing to infinity if the region is finite. The same is true for the infinite region, if the auxiliary potential v increases at infinity beyond all bounds. Conversely, /469 if the auxiliary potential has a finite lower limit at infinity, the spectrum below this value will be discrete.

These criteria correspond to one portion of the criteria established by Weyl in a theory of differential equations with singularities; however, they can be proved in a manner independent on the number of variables.

The present paper contains Part I of these investigations; Part II will be published in one of the next issues.

PART I

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SPECTRAL THEORY OF SEMIBOUNDED OPERATORS

1. Basic Concepts

We will first assemble a few well-known (Bibl.7.2) basic concepts and theorems on forms and operators, using a system of notations suitable for our own purposes so as to be independent of the remaining literature.

A space $\mathfrak{X}$, $\mathfrak{H}$, $\mathfrak{G}$, $\mathfrak{F}$ of elements x, h, g, f is to mean always a real linear space with - unless stated differently - at least denumerably infinitely many linearly independent elements.

A bilinear form coordinates, to each pair of elements x, x_1, a real number which is linear in x and x_1; we denote this* by

$$x_1 A\, x.$$

We always assume

$$x_1 A\, x = x A\, x_1$$

i.e., that **A** be symmetric.

Let a quadratic form xAx, consisting of such a bilinear form, be denoted as "never negative", if

$$x A\, x \geqq 0$$

is valid; for such forms - as follows already from the known method of reasoning - the following Schwarz inequality is valid: /471

$$x_1 A\, x \leqq \sqrt{(x_1 A\, x_1)}\, \sqrt{(x A\, x)}.$$

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* This symbolism is patterned after that given by Dirac. Our entire theory can be developed also in the complex space, with only minor and completely conventional modifications.

This expression is called positive-definite only if $x = 0$ always follows from $xAx = 0$.

If, in a space $\mathfrak{X}$, a metric $|x| = \sqrt{xXx}$ has been introduced by a positive-definite "measure" xXx, then it will be possible to explain the density of a set in $\mathfrak{X}$ and the convergence of a sequence x. We will formulate this as follows:

A sequence x converges strongly (X)

I) "in itself" if $|x_\nu - x_\mu| \to 0$, $\nu, \mu \to \infty$

II) to x, if $|x_\nu - x| \to 0$, $\nu \to \infty$

is valid.

If the space is separable (i.e., if it contains a denumerable dense set) and closed (also complete), meaning that a limiting element exists for each convergent sequence, then this space becomes known as an (abstract) Hilbert space.

Below, we will use a Hilbert space $\mathfrak{H}$ of elements h, with the measure form* H.

It is of importance for our treatment that frequently subspaces of $\mathfrak{H}$, such as $\mathfrak{G}$, are used as Hilbert spaces with a different measure such as G and that then both convergence and density are referred to these. In that case, we speak for example of convergence (G) and of G-density.

Subspaces $\mathfrak{F}$, $\mathfrak{G}$, ... of $\mathfrak{H}$, in which forms and operators are explained, will always be assumed as H-dense in $\mathfrak{H}$.

An operator A explained in $\mathfrak{F} \leq \mathfrak{H}$, coordinates an element h from $\mathfrak{H}$ with each element f from $\mathfrak{F}$.

To each operator A in $\mathfrak{F}$, there "belongs" the form A in $\mathfrak{F}$:

$$f_1 A f = (f_1 H A f).$$

* The usual notation is obtained on replacing H by a comma.

The operator is known as symmetric if it constitutes the corresponding form; unless stated otherwise, operators will always be assumed as symmetric and linear.

An operator B in $\mathfrak{F}$ is denoted as bounded* if this is its form B; there exist two real numbers $\underline{\beta};\beta$, so that $\underline{\beta}(fHf) \leq (fBf) \leq \overline{\beta}(fHf)$ is valid here.

The next simple class is represented by the semibounded forms. A form **G**, defined in $\mathfrak{G} \leq \mathfrak{H}$, is known as positive-semibounded** (downward) with the (lower) bound $\underline{\gamma}$ if a bound $\underline{\gamma} > 0$ exists such that, for all elements g from $\mathfrak{G}$, /472

$$gGg \geqq \underline{\gamma}(g\,H\,g)$$

Correspondingly, an operator is known as positive-semibounded if it is the corresponding form.

Of importance for what follows is the property of the "state of closure" that positive-semibounded forms may possess.

A positive-semibounded form G is known as closed in $\mathfrak{G}$ if the space $\mathfrak{G}$ is closed, with gGg as measure. Then $\mathfrak{G}$ is also a Hilbert space with **G** as measure***.

For (not necessarily semibounded) operators A, defined in dense subsets $\mathfrak{F}$

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* The non-symmetric operator S is denoted as bounded if the following is valid for f, f_1 from $\mathfrak{F}$:

$$|f\,H\,S\,f_1| < C|f|\,|f_1|.$$

** Without the assumption $\underline{\gamma} > 0$, the form would be only semibounded but can always, by addition of $(1 - \underline{\gamma})$ (gHg), be changed into a positive-semibounded form.

*** It is easy to construct a denumerable G-dense subset $(\mathfrak{G})$ of $\mathfrak{G}$. First – separability of $\mathfrak{H}$ – a denumerable H-dense set $(\mathfrak{M})$ exists in each subset $\mathfrak{M}$ of $\mathfrak{H}$. Let us now form the sequence of subsets $\mathfrak{G}_n$ of $\mathfrak{G}$, characterized by the condition $gGg \leq n(gHg)$. Each element g belongs to one of these. The set $(\mathfrak{G}_n)$, i.e., the denumerable H-dense subset of $\mathfrak{G}_n$, is then also G-dense in $\mathfrak{G}_n$ and the sum of the $(\mathfrak{G}_n)$ will have the property desired of $(\mathfrak{G})$.

of $\mathfrak{H}$, v.Neumann (Bibl.7.3) introduced the concept of the state of closure on which the above concept is patterned. His formulation is equivalent to the following [similar to that mentioned by him (Bibl.7.6)]:

An operator A in $\mathfrak{F}$ is known as closed if the space $\mathfrak{F}$ is closed, with

$$A\,|\,H\,A\,| + |\,H\,| \quad (\text{in short } A^2 + H)$$

as measure.

2. Continuation by Closure

The differential operators of prime interest here usually are not given in closed spaces and are themselves not closed, whereas this property is presupposed in the general spectral theory. Already for this reason it is of interest to investigate whether spaces, operators, and forms can be continued to closed types in expanded definition domains.

Let a space $\mathfrak{H}'$ of elements h and with the measure H be given. To continue the space $\mathfrak{H}'$ to a space $\mathfrak{H}$ involves: adjoining additional - also denoted by h - /473
(ideal) elements, followed by, together with the elements from $\mathfrak{H}'$, addition and multiplication with real numbers, and then defining the form H positive-definite.

Then the following theorem applies:

Theorem 1. If the space $\mathfrak{H}'$ with the measure H is separable, it can be continued to a closed, i.e., a Hilbert, space $\mathfrak{H}$, but only by one single process* such that $\mathfrak{H}'$ is dense in $\mathfrak{H}$.

Proof. Let each self-converging sequence h_ν from $\mathfrak{H}'$

$$|h_\nu - h_\mu| \to 0,$$

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* This means that two continuations can be conformally mapped onto each other in such a manner that multiplication with real numbers, addition, and values of the form H can be simultaneously mapped.

be coordinated with an ideal limiting element h, unless there is already a limiting element of $\mathfrak{H}'$ in existence. Let the same limiting element be coordinated with two such sequences $h_{1\nu}$, $h_{2\nu}$ for which $|h_{1\nu} - h_{2\nu}| \to 0$. If two sequences $h_{1\nu}$, $h_{2\nu}$ possess limiting elements h_1, h_2 (from $\mathfrak{H}'$ or ideal types), then $h_{1\nu} H h_{2\nu}$ will have a limiting value, which is defined here as $h_1 H h_2$. It is easy to demonstrate that H is then bilinear and positive-definite in the entire space $\mathfrak{H}$ and that any continuation, stipulated in theorem 1, can be produced in this manner.

Let an operator A be defined in $\mathfrak{F}' \leq \mathfrak{H}$. To continue the operator A in $\mathfrak{F}'$ in a subspace $\mathfrak{F}$ of $\mathfrak{H}$ ($\mathfrak{F}' < \mathfrak{F} \leq \mathfrak{H}$) means to define it, for the elements of $\mathfrak{F}$ not located in $\mathfrak{F}'$, such that it will be linear and symmetric in all of $\mathfrak{F}$.

The continuation of a form can be explained in an entirely similar manner.

In that case, the following theorems apply:

<u>Theorem 2</u>. A bounded operator B or a bounded form B, explained only in a dense subspace of $\mathfrak{H}$, can be uniquely continued with the same bounds β in $\mathfrak{H}$*.

<u>Theorem 3</u>. An operator G, explained in the space $\mathfrak{F}'$ of the elements f, is assumed to lead to a positive-semibounded form G with the bound $\underline{\gamma} > 0$

$$/HG/ = /G/ \geqq \underline{\gamma}(/H/).$$

Then, a subspace $\mathfrak{G}$ of $\mathfrak{H}$ containing $\mathfrak{F}'$ ($\mathfrak{F}' \leq \mathfrak{G} \leq \mathfrak{H}$) exists in which the form **G** can be continued to a closed form with the same bound $\underline{\gamma}$. $\mathfrak{G}$ and **G** in $\mathfrak{G}$ are uniquely defined if $\mathfrak{F}'$ is to be **G**-dense in $\mathfrak{G}$.

Let **G** in $\mathfrak{G}$ be the closed form belonging to **G** in $\mathfrak{F}'$ or G in $\mathfrak{F}'$.

Before proving theorem 3, we should mention another theorem. /474

<u>Theorem 4</u>. To each operator A in $\mathfrak{F}'$, there exists a subspace $\mathfrak{F}$ of $\mathfrak{H}$,

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* Similar statements apply also to non-symmetric operators.

containing $\mathfrak{F}'$, in which a closed continuation of A can be defined. Here, $\mathfrak{F}$ and A in $\mathfrak{F}$ are uniquely defined if $\mathfrak{F}'$ is to lie dense in $\mathfrak{F}$, with $A^2 + H$ as measure.

A in $\mathfrak{F}$ is denoted as the closed operator of A in $\mathfrak{F}'$.

Theorem 4 and its simple proof were given by v.Neumann (Bibl.7.3).

Remark on theorems 3 and 4. The operator G in $\mathfrak{F}'$ is assumed to lead to the form **G** in $\mathfrak{F}'$. Let G in $\mathfrak{F}$ be the corresponding closed operator and **G** in $\mathfrak{G}$ the corresponding closed form. Then, $\mathfrak{F}$ comes to lie in $\mathfrak{G}$.

Let us first prove this remark, and then prove theorem 4 and finally theorem 3.

Proof of the remark. If f is located in $\mathfrak{F}$, it can be H-approximated by a series f_ν for which also Gf_ν will H-converge so that we also have

$$(f_\nu - f_\mu)\, H G\, (f_\nu - f_\mu) \to 0 \quad \text{for} \quad \nu, \mu \to \infty,$$

i.e.,

$$(f_\nu - f_\mu)\, \mathbf{G}\, (f_\nu - f_\mu) \to 0,$$

which means that f_ν also converges with respect to **G**; the limiting element f, according to theorem 3, must also lie in $\mathfrak{G}$.

Next, let us briefly review the proof of theorem 4 according to v.Neumann. If, for a sequence f_ν, the following form converges

$$(f_\nu - f_\mu)\, H\, (f_\nu - f_\mu) + A\,(f_\nu - f_\mu)\, H A\, (f_\nu - f_\mu) \to 0 \quad \text{at} \quad \nu, \mu \to \infty,$$

there will be limiting elements f_0 and h_0 so that

$$f_\nu \to f_0, \quad A f_\nu \to h_0 \;(H).$$

All possibly resulting elements f_0 will constitute the obviously linear space $\mathfrak{F}$. If two such sequences $f_{1\nu}$, $f_{2\nu}$ possess the same limiting element $f_{10} = f_{20}$, then also $Af_{1\nu}$ and $Af_{2\nu}$ will have the same limiting element since, for all h from $\mathfrak{H}$, the following is valid:

$$h H (f_{1\nu} - f_{2\nu}) \to 0,$$

and thus also for

$$h = A f,$$

where f represents all elements from $\mathfrak{F}'$, i.e.,

$$A f H (f_{1\nu} - f_{2\nu}) = f H A (f_{1\nu} - f_{2\nu}) \to 0 \quad \text{at} \quad \nu, \mu \to \infty$$

and, consequently,

$$f H (h_{10} - h_{20}) = 0;$$

Since $\mathfrak{F}'$ is densely located, it follows that $h_{10} = h_{20}$. It is now possible /475
to define the operator A in all of $\mathfrak{F}$, by $Af_0 = h_0$; obviously, this operator remains linear and symmetric.

The proof of theorem 3 is conducted similarly. Incidentally, not every positive-semibounded form can be continued to a closed form; for this, an auxiliary condition - for example the condition that this form belong to an operator - is required.

For proving theorem 3, we will use theorem 2 for closing off the space $\mathfrak{F}'$, with the measure G, by adjunction of first ideal elements, yielding a space $\mathfrak{G}$ of elements g with the measure G. Each sequence f_ν of elements from $\mathfrak{F}'$, which is self-convergent with G as measure, will also converge with H as measure, because of

$$\underline{\gamma} (f H f) \leq f G f,$$

which means that it has a limiting element h from $\mathfrak{H}$. However, at first glance it does not seem impossible that two different limiting elements g_1 and g_2 might correspond to two sequences $f_{1\nu}$ and $f_{2\nu}$, while the limiting elements $h_1 = h_2$ are identical. We will demonstrate that this does not occur. From the assumption that an operator G belongs to **G**, we can conclude that:

<u>Lemma 3.1</u>. If, for a sequence f_ν from $\mathfrak{F}'$,

$$h H f_\nu \to 0 \qquad \text{for all h from } \mathfrak{H}$$

is valid, it follows that*

$$f G f_\nu \to 0 \qquad \text{for all f from } \mathfrak{F}'$$

This is so, since

$$f G f_\nu = G f H f_\nu \to 0.$$

From lemma 3.1 it then follows readily:

Lemma 3.2. If, for a sequence f_ν from $\mathfrak{F}'$,

$$(f_\nu - f_\mu) G (f_\nu - f_\mu) \to 0 \quad \text{for} \quad \nu, \mu \to \infty$$

is valid and

$$f_\nu H f_\nu \to 0 \quad \text{for} \quad \nu \to \infty,$$

then we also have

$$f_\nu G f_\nu \to 0 \quad \text{for} \quad \nu \to \infty.$$

This is so since, for such a sequence, the prerequisite of lemma (3.1) is /476 satisfied; thus, $f_\mu G f_\nu \to 0$ at fixed μ. Next, the Schwarz inequality

$$|f_\mu G f_\mu - f_\mu G f_\nu| \leqq \sqrt{(f_\nu - f_\mu) G (f_\nu - f_\mu)} \sqrt{f_\mu G f_\mu}$$

is used, Here, the lower limit of the right-hand side, as $\nu \to \infty$, tends to zero with increasing μ. The lower limit of the left-hand side for $\nu \to \infty$ is $f_\mu G f_\mu$; consequently, also $f_\mu G f_\mu \to 0$ with increasing μ.

This proves directly: If $f_{1\nu}$ and $f_{2\nu}$ are two sequences from $\mathfrak{F}'$ which define two limiting elements g_1 and g_2 and two limiting elements h_1, h_2 and if $h_1 = h_2$, then the prerequisites of lemma (3.2) are satisfied for the difference $f_\nu = f_{1\nu} - f_{2\nu}$. Consequently, we have $(f_{1\nu} - f_{2\nu})G(f_{1\nu} - f_{2\nu}) \to 0$, from which it

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* In addition, if $f_\nu G f_\nu$ remains bounded, we have $g G f_\nu \to 0$ also for all g from $\mathfrak{G}$ since $\mathfrak{F}'$ is G-dense in $\mathfrak{G}$. Then, lemma 3.1 means: If such a sequence f_ν converges weakly in $\mathfrak{H}$ with H as measure, then it must also converge in $\mathfrak{G}$ with G as measure. This property (3.1) is also equivalent to the semicontinuity of G in the case of weak G convergence; the next proof is nothing else but an abstract formulation of frequently used semicontinuity proofs; see also Courant (Bibl.2.4) for the method of reasoning used.

follows that $(g_1 - g_2)G(g_1 - g_2) = 0$ and thus also $g_1 = g_2$.

If $g_1 = g_2$ is a real element from $\mathfrak{F}'$, then $g_1 = g_2 = h_1 = h_2$, since the sequence $f_{1\nu}$ converges also with respect to H toward g_1. Conversely, if $g_1 = g_2$ is an ideal element from $\mathfrak{G}$, it will be identified by $h_1 = h_2$. However, this will change $\mathfrak{G}$ into a subspace of $\mathfrak{H}$.

3. Operator of a Form

Whereas a certain form belongs directly to each operator, the opposite is the case only to a limited extent. According to Fr.Riesz (Bibl.9.2), the following applies:

Theorem 5. To each bounded form B in $\mathfrak{H}$ there corresponds a bounded operator B in $\mathfrak{H}$, so that we have

$$h\,B\,h = h\,H\,B\,h.$$

Proof. Let is be permitted to give a simple proof which does not refer to representation by an orthogonal system.

Let us pose the minimum problem, namely, that of coordinating, to each element k_1 from $\mathfrak{H}$, such an element $h = k_0$ for which

$$J[h] = h\,H\,h - 2\,h\,B\,k_1$$

becomes as small as possible. It is certain that J[h] is bounded downward since B is bounded. Consequently, a lower bound d and a minimal sequence h_ν exist.

For these, it follows that

$$d_\nu = h_\nu\,H\,h_\nu - 2\,h_\nu\,B\,k_1 \to d$$

and, from

$$(h_\nu + \varepsilon\,h)\,H\,(h_\nu + \varepsilon\,h) - 2\,(h_\nu + \varepsilon\,h)\,B\,k_1 \geqq d,$$

that

$$(d_\nu - d) + 2\,\varepsilon\,(h\,H\,h_\nu - h\,B\,k_1) + \varepsilon^2\,h\,H\,h \geqq 0,$$

meaning that it is never negative in ϵ, so that

$$\sqrt{(d_\nu - d)\, h\, H\, h} \geqq |h\, H\, h_\nu - h\, B\, k_1|$$

is valid for all h from $\mathfrak{H}$. From this it follows first that

$$h\, H\, h_\nu - h\, B\, k_1 \to 0;$$

and then, by setting $h = h_\nu - h_\mu$ and permuting ν with μ,

$$\begin{aligned}(h_\nu - h_\mu)\, H\, (h_\nu - h_\mu) = |&(h_\nu - h_\mu)\, H\, h_\nu - (h_\nu - h_\mu)\, B\, k_1 \\ &+ (h_\mu - h_\nu)\, H\, h_\mu - (h_\mu - h_\nu)\, B\, k_1| \\ \leqq &\left(\sqrt{d_\nu - d} + \sqrt{d_\mu - d}\right) \sqrt{(h_\nu - h_\mu)\, H\, (h_\nu - h_\mu)}\end{aligned}$$

and, consequently,

$$(h_\nu - h_\mu)\, H\, (h_\nu - h_\mu) \leqq \left(\sqrt{d_\nu - d} + \sqrt{d_\mu - d}\right)^2 \to 0.$$

However, this means that h_ν converges toward a limiting element k_0. For this, with each h from $\mathfrak{H}$, we obtain

$$h\, H\, k_0 - h\, B\, k_1 = 0. \tag{*}$$

The coordination of k_0 with k_1 is denoted here as operator B, i.e., we set

$$k_0 = B\, k_1.$$

Then, we will obtain:

1) B is unique since the difference of two k_0, belonging to the same k_1, would have to be orthogonal on all h, according to eq.(*).

2) B is linear since the sum of two k_0 satisfies the relation (*) for the sum of the k_1. However, its existence is characteristic for the minimum property since

$$\begin{aligned}(k_0 + h)\, H\, (k_0 + h) &- 2\,(k_0 + h)\, B\, k_1 \\ &\geqq k_0\, H\, k_0 - 2\, k_0\, B\, k_1 = d\end{aligned}$$

follows from it.

3) B belongs to the form B since eq.(*), for $k_1 = h$, is transformed

into

$$h H B h - h B h = 0.$$

From this it follows directly: 4) B is symmetric and 5) B is bounded. As a new theorem, we can formulate the following:

<u>Theorem 6</u>. For a closed form G in $\mathfrak{G} \leq \mathfrak{H}$, semibounded by $\underline{\gamma} > 0$, a bounded operator B exists such that

$$g G B h = g H h$$

is valid for g from $\mathfrak{G}$, h from $\mathfrak{H}$. We have

$$0 \leqq h H B h \leqq \frac{1}{\gamma} (h H h)$$

and the value domain of B lies in $\mathfrak{G}$.

This operator B will later be proved to be the reciprocal of an operator /478 belonging to G.

<u>Proof</u>. In the Hilbert space $\mathfrak{G}$, H is a bounded form

$$g H g \leqq \frac{1}{\gamma} (g G g);$$

which means that according to theorem 5 a bounded operator B, explained in $\mathfrak{G}$, exists for which

$$g H h = g G B h \tag{**}$$

is first valid for all h from $\mathfrak{G}$. On setting g = Bh, with h from $\mathfrak{G}$, we obtain

$$(Bh\, G\, Bh)^2 = (Bh\, H\, h)^2 \leqq (Bh\, H\, Bh)\,(h\, H\, h) \leqq \frac{1}{\gamma} (Bh\, G\, Bh)(h\, H\, h)$$

or

$$(Bh\, G\, Bh) \leqq \frac{1}{\gamma} (h\, H\, h),$$

i.e.,

$$(h\, H\, Bh) \leqq \frac{1}{\gamma} (h\, H\, h). \tag{***}$$

This relation, which is primarily valid for h in $\mathfrak{G}$, demonstrates that B is also H-bounded and thus can be continued to $\mathfrak{H}$. From eq.(†) and the assumed state of closure of the form G it is concluded that Bh always lies in $\mathfrak{G}$ and that, accordingly, the relations (**) and (***) hold also for all h from $\mathfrak{H}$.

Theorem 6 could have been proved also by directly applying the minimum requirement*, without referring to theorem 5.

From theorem 6, it follows directly that:

<u>Theorem 7</u>. To a closed positive-semibounded form G in $\mathfrak{G} \leq \mathfrak{H}$ there will exist, in a G-dense subspace $\mathfrak{F}_1 \leq \mathfrak{G}$, an operator which "belongs" because of

$$g H G f = g G f, \qquad f \text{ from } \mathfrak{F}_1$$

and whose value range is $\mathfrak{H}$. Let us denote this as "maximally corresponding" operator for G.

<u>Proof</u>. Let us, according to theorem 6, construct the bounded operator B; let the range of values of f = Bh be $\mathfrak{F}_1$. Then $\mathfrak{F}_1$ is located in $\mathfrak{G}$. Now, only one h corresponds to each f from $\mathfrak{F}_1$, since it follows from Bh = 0, because of

$$g G B h = g H h = 0 \quad \text{for all } g \text{ from } \mathfrak{G}$$

that also h = 0. Thus, the reciprocal of B is uniquely defined in $\mathfrak{F}_1$; this /479
will be denoted by G, i.e., it is assumed the Bh = f is equivalent to h = Gf.

The space $\mathfrak{F}_1$ is G-dense in $\mathfrak{G}$, since otherwise an element $g_0 \neq 0$ from $\mathfrak{G}$ would exist which would be G-orthogonal on $\mathfrak{F}_1$:

$$g_0 G f = 0; \quad \text{thus also} \quad g_0 G B g_0 = g_0 H g_0 = 0, \quad \text{i.e.} \quad g_0 = 0.$$

From this it follows that:

1) G is linear since G is unique and B is linear.

* Theorem 6 is related with the theorem by Toeplitz (Bibl.11) on the limiting resolvent of positive-definite forms of infinitely many variables, a theorem which can be proved in a similar manner but more conveniently than with the Jacobi transformation (see Mathem. Annalen, Vol.109, pp.254-256). Toeplitz's theorem also formed the starting point for Wintner's theory (Bibl.13) of semi-bounded matrices.

2) G is symmetric since B is symmetric.

Auxiliary theorem 7. If G' in $\mathfrak{F}' \leq \mathfrak{G}$ is an operator belonging to the form G, i.e., an operator for which, with f' from $\mathfrak{F}'$ and g from $\mathfrak{G}$, the following is valid

$$g\,G\,f' = g\,H\,G'\,f',$$

then the "maximally corresponding" operator G in $\mathfrak{F}_1$ is a continuation of G' in $\mathfrak{F}'$.

Proof. With the operator B of theorem 6, let us form the element $BG'f'$ from $\mathfrak{F}_1$ for the element f' from $\mathfrak{F}'$. Then, the following is valid for all g from $\mathfrak{G}$:

$$g\,G\,B\,G'\,f' = g\,H\,G'\,f' = g\,G\,f'; \text{ thus } B\,G'\,f' = f'; \text{ i.e., } f' \text{ in } \mathfrak{F}_1.$$

4. Selfadjoint (Hypermaximal) Operators

As demonstrated by v.Neumann (Bibl.7.2), a spectral analysis is not possible for each closed symmetric operator; for this (according to E.Schmidt), the condition of hypermaximality (Bibl.7.3) would have to be satisfied. The same condition was stipulated by Stone (Bibl.10.1) as self-adjointness of his spectral theory.

Definition. Let a symmetric operator A be explained in $\mathfrak{F} \leq \mathfrak{H}$. Let A in $\mathfrak{F}$ be selfadjoint if the following condition is satisfied:

If an element h_0 from $\mathfrak{H}$ is coordinated with another h_1 from $\mathfrak{H}$, so that the following is valid for all f from $\mathfrak{F}$

$$h_0\,H\,A\,f = h_1\,H\,f,$$

then h_0 is located in $\mathfrak{F}$ and we have $Ah_0 = h_1$.

Obviously, a selfadjoint operator is always closed.

It is generally known that:

Theorem 8. A bounded operator is selfadjoint.

To this, we juxtapose:

Theorem 9*. The operator G in $\mathfrak{F}_1 \leq \mathfrak{G}$, maximally belonging to a closed /480
form G in $\mathfrak{G}$, is selfadjoint. We then speak of a corresponding selfadjoint operator.

Proof. According to v.Neumann (Bibl.7.4) each symmetric operator with a value range $\mathfrak{H}$ is selfadjoint. This is so since then an f_0 exists for h_1, so that $Af_0 = h_1$; consequently,

$$h_0 H A f = A f_0 H f = f_0 H A f,$$

is valid and, since the Af traverse the entire $\mathfrak{H}$, it follows that $f_0 = h_0$. From this follows the assertion according to theorem 7.

It can now be demonstrated that, for positive-semibounded operators, the self-adjointness follows already from weaker conditions that are easy to check for our differential operators.

Let G in $\mathfrak{F}$ be a closed positive-semibounded operator; let gGg in $\mathfrak{G}$ be the pertinent closed form according to theorem 3; let G in $\mathfrak{F}_1$ be the corresponding selfadjoint operator according to theorems 7 and 9. Then, we can introduce the "iterated" spaces

$$\mathfrak{G}_1,\ \mathfrak{F}_2,\ \mathfrak{G}_2,\ \mathfrak{F}_3,\ \mathfrak{G}_3,\ \ldots$$

which consist of all elements f of $\mathfrak{F}_1$ for which, for example, Gf is located in

$$\mathfrak{G},\ \mathfrak{F}_1,\ \mathfrak{G}_1,\ \mathfrak{F}_2,\ \mathfrak{G}_2,\ \ldots$$

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* Theorem 9 also furnishes the proof for a conjecture by v.Neumann (Bibl.7.5): A positive-semibounded operator can be continued to a selfadjoint operator with the same lower bound. This is supposedly so because of the fact that, if this operator G lies in $\mathfrak{F}'$ then the corresponding form G, according to theorem 3, is continued to a closed form; the corresponding selfadjoint operator G in $\mathfrak{F}_1$, according to the auxiliary theorem 7, is a continuation of G in $\mathfrak{F}'$; that this operator belongs to the same lower bound follows from the fact that this is valid for the form G. The only undecided point is whether such a continuation can be obtained also in a different manner.

We then have:

Theorem 10. It is known that $\mathfrak{F} = \mathfrak{F}_1$, i.e., that G in $\mathfrak{F}$ is selfadjoint if, for any n (n = 1, 2, 3, ...), the following is valid:

$$\mathfrak{F}_n \text{ lies in } \mathfrak{F} \qquad (\mathfrak{F}_n)$$

or

$$\mathfrak{G}_n \text{ lies in } \mathfrak{F} \qquad (\mathfrak{G}_n)$$

Proof. Since $\mathfrak{F}_{n+1}$ lies in $\mathfrak{G}_n$, it is sufficient to prove theorem 10 under the assumption $(\mathfrak{F}_n)$. Since, according to the auxiliary theorem 7 and $(\mathfrak{F}_n)$, the operators G explained in $\mathfrak{F}_1$, $\mathfrak{F}$, $\mathfrak{F}_n$ are mutual continuities and since $\mathfrak{F}$ is closed, it is sufficient to demonstrate:

The closed operator of G in $\mathfrak{F}_n$ is G in $\mathfrak{F}_1$. For the proof, it must be /481 taken into consideration that $\mathfrak{F}_n$ constitutes the value range of $B^n h$ where B is the reciprocal of G in $\mathfrak{F}_1$ according to theorem 6. From this it follows that $\mathfrak{F}_n$ is dense in $\mathfrak{H}$; if this were not the case, an element $h_0 \neq 0$ with $0 = h_0 H B^n h = B^n h_0 H h$ would exist for all h, from which it would follow that $B^n h_0 = B^{n-1} h_0 = \ldots = h_0 = 0$. Let now f_1 be an element of $\mathfrak{F}_1$, so that a sequence f^ν from $\mathfrak{F}_{n+1}$ exists for which $f^\nu \to Gf_1$. Then we also have $Bf^\nu \to f_1$ and $G(Bf^\nu) \to Gf_1$, i.e., the closed operator to G in $\mathfrak{F}_n$ is explained for f_1 and yields Gf_1.

5. Spectral Analysis

Before formulation and proof of the spectral theorem of semibounded forms, let us give a number of known concepts and correlations on the spectral family, in a system of notation and arrangement suitable for our purposes.

According to v.Neumann (Bibl.7.2), any representation of the spectral resolution must be based on "projection operators or individual operators".

A projection operator P is an operator defined in $\mathfrak{H}$ which satisfies the relation

$$P^2 = P$$

Obviously, 1 - P is such an operator.

Let the elements Ph and (1 - P)h be the eigenelements and antielements of P; let their value domains $\mathfrak{P}$ and $\mathfrak{H} \ominus \mathfrak{P}$ be the eigenspace and antispace of P.

The requirement of symmetry

$$P h_1 H h_2 = h_1 H P h_2$$

for h_1, h_2 from $\mathfrak{H}$ is equivalent to

$$P h_1 H (1 - P) h_2 = 0$$

i.e., to the fact that eigenelements and antielements or eigenspace and antispace of P are orthogonal.

Equivalent is also the identity

$$h H h = P h H P h + (1 - P) h H (1 - P) h.$$

For the form P, belonging to a symmetric P and constituting the "single form", the following is valid:

$$0 \leqq h P h \leqq h H h,$$

where equality (for all h) exists only for P = 0 resp. P = 1. Specifically, the boundedness of P is demonstrated in this manner from the symmetry.

The properties, relating to the unitary form H, can be analogously explained also for other forms. We say that P is symmetric with respect to the form A in $\mathfrak{F}$ or, briefly, A-symmetric in $\mathfrak{F}$ provided that not only f but also Pf lies in $\mathfrak{F}$ and the equivalent identities for f, f_1, f_2 from $\mathfrak{F}$ exist: /482

$$P f_1 A f_2 = f_1 A P f_2,$$
$$P f A (1 - P) f = 0,$$
$$f A f = P f A P f + (1 - P) f A (1 - P) f.$$

Then, the following holds:

$$0 \leqq fAPf \leqq fAf.$$

A projection operator P can be permuted with an operator A in $\mathfrak{F}$ if, together with f, also Pf is located in $\mathfrak{F}$ and if

$$APf = PAf$$

is valid. The operator P is permutable with A in $\mathfrak{F}$ as soon as P is not only symmetric with respect to H but also with respect to the form fAf in $\mathfrak{F}$, belonging to A.

Below, we will always assume that P is symmetric (i.e., H-symmetric in $\mathfrak{H}$).

The fact that, between two projection operators P_1, P_2, the following relation exists for all h from $\mathfrak{H}$.

$$hP_1h \geqq hP_2h$$

is equivalent with the fact that the eigenspace of P_1 contains that of P_2 and the antispace of P_2 that of P_1. Expressed in formulas, this reads

$$P_2 P_1 = P_2, \quad (1 - P_2)(1 - P_1) = (1 - P_1) \quad \text{or} \quad P_1 P_2 = P_2.$$

Then, also $P_1 - P_2$ is a projection operator.

The spectral family is a family of projection operators that depend on a real parameter α in such a manner that the pertaining forms are monotonic. At the points α at which these forms may become discontinuous, two projection operators can be attained as limiting values from above and from below. It is suggested to imagine each value of α as coordinated with two projection operators whose forms are continuous from top to bottom. For more convenient notation, we will designate the projection operators by the symbols α^+ and α^- (instead of $\alpha + 0$ and $\alpha - 0$). This motivates the following explanation:

To each real number α, let the symbols α^+, α^- be coordinated; in general, let α^+ and α^- be denoted by $\alpha^\bullet$; for $\alpha = \infty$ and $\alpha = -\infty$, let $\alpha^\bullet = \infty^-$ and $\alpha^\bullet = -\infty^+$ be introduced. We set $\alpha_1^\bullet < \alpha_2^+$ if $\alpha_1 < \alpha_2$ and, in addition, $\alpha^- < \alpha^+$.

The interval

$$\Delta\alpha = (\alpha_1^\bullet, \alpha_2^\bullet)$$

is assumed to contain each point α for whose two symbols $\alpha^\bullet$ the following is valid: $\alpha_1^\bullet \leq \alpha^\bullet \leq \alpha_2^\bullet$; this automatically includes intervals with and without end points, (α^-, α^+) contains only the point α.

Now, let a projection operator $P_{\alpha^\bullet}$ be coordinated with each $\alpha^\bullet$ in such /483 a manner that, for each h from $\mathfrak{H}$ with the corresponding forms $hP_{\alpha^\bullet}h$, the following holds:

1) If

$$\alpha_1^\bullet < \alpha_2^\bullet, \text{ then } hP_{\alpha_1^\bullet}h \leqq hP_{\alpha_2^\bullet}h.$$
$$\nu \to \infty$$

2) If, as $\nu \to \infty$,

$$\alpha_\nu \downarrow \alpha, \text{ then } hP_{\alpha_\nu^\bullet}h \downarrow hP_{\alpha^+}h,$$
$$\alpha_\nu \uparrow \alpha, \text{ then } hP_{\alpha_\nu^\bullet}h \uparrow hP_{\alpha^-}h.$$

Then, $P_{\alpha^\bullet}$ is a spectral family.

The spectral family is complete if

$$P_{-\infty^+} = 0, \quad P_{\infty^-} = 1$$

is valid. The difference operator

$$P_{\Delta\alpha} = P(\alpha_1^\bullet, \alpha_2^\bullet) = P_{\alpha_2^\bullet} - P_{\alpha_1^\bullet}$$

of the interval $\Delta\alpha = (\alpha_1^\bullet, \alpha_2^\bullet)$ again is a projection operator. It represents the skip operator

$$P(\alpha^-, \alpha^+) \neq 0$$

only for an at most denumerable set of values α, known as point eigenvalues.

Let the spectral theorem for bounded forms - as is suitable for our purpose - be postulated as follows:

Theorem 11. Let B in $\mathfrak{H}$ be a bounded form with the bounds $\underline{\beta}$, $\overline{\beta}$; let B in $\mathfrak{H}$ be the corresponding operator. Then, there will be exactly one complete spectral family $Q_{\beta^\bullet}$, so that the following holds:

1) $Q_{\beta^\bullet}$ is symmetric with respect to the form B and with respect to H.

2) The two "eigenvalue inequalities"

$$\beta_2 (h\, H\, Q_{\Delta\beta}\, h) \leqq (h\, B\, Q_{\Delta\beta}\, h) \leqq \beta_1 (h\, H\, Q_{\Delta\beta}\, h)$$

exist, with $\Delta\beta = (\beta_2^\bullet, \beta_1^\bullet)$ for all h from $\mathfrak{H}$.

Auxiliary theorem 11.1. We have $Q_{\underline{\beta}^-} = 0$, $Q_{\overline{\beta}^+} = 1$.

Auxiliary theorem 11.2. The operator B in $\mathfrak{H}$ can be permuted with $Q_{\beta^\bullet}$.

The proof of theorem 11 and of the auxiliary theorems is readily obtained from the conventional formulations of the spectral theorem [see specifically Fr.Riesz (Bibl.9.3)].

The spectral theorem for closed positive-semibounded forms reads as follows:

Theorem 12. Let G in $\mathfrak{G}$ be a closed form, semibounded downward by $\underline{\gamma} > 0$, and let G in $\mathfrak{F}_1 \leq \mathfrak{G}^-$ be the corresponding selfadjoint operator. Then, exactly one complete spectral family $R_{\gamma^\bullet}$ will exist, so that we have: /484

1) $R_{\gamma^\bullet}$ is symmetric with respect to the form G in $\mathfrak{G}$ and with respect to H in $\mathfrak{H}$.

2) The two "eigenvalue inequalities"

$$\gamma_1 (g\, H\, R_{\Delta\gamma}\, g) \leqq (g\, G\, R_{\Delta\gamma}\, g) \leqq \gamma_2 (g\, H\, R_{\Delta\gamma}\, g)$$

exist, with $\Delta\gamma = (\gamma_1^\bullet, \gamma_2^\bullet)$ for all g from $\mathfrak{G}$.

Auxiliary theorem 12.1. We have $R_{\underline{\gamma}^-} = 0$, $R_{\infty^-} = 1$.

Auxiliary theorem 12.2. The operator G in $\mathfrak{F}_1$ can be permuted with $R_{\gamma^\bullet}$.

Auxiliary theorem 12.3. The eigenelements of the differential operators $R_{\Delta\gamma}$ of finite intervals $\Delta\gamma$ are located in $\mathfrak{F}_1$.

We will say that the spectral family $R_{\gamma}\cdot$ furnishes the spectral resolution of G in $\mathfrak{G}$ and of G in $\mathfrak{F}_1$.

Proof. Theorem 12 is a simple logical consequence of theorem 11. This is so since, in the Hilbert space $\mathfrak{G}$, the form H will become a form bounded with respect to $\mathfrak{G}$, with the bounds 0 and $\frac{1}{\underline{\gamma}} = \bar{\beta}$; here theorem 11 can be applied, furnishing a spectral family $Q_{\beta}\cdot$. Then, we set $\beta = \frac{1}{\gamma}$ for $\beta > 0$. In this manner,

$$R_{\gamma^+} = 1 - Q_{\beta^-}, \quad R_{\gamma^-} = 1 - Q_{\beta^+}$$

will explain also a spectral family. This family is complete since, first, the auxiliary theorem 11.1 indicates that we have $Q_{\bar{\beta}^+} = 1$, i.e., $R_{\underline{\gamma}^-} = 0$. Secondly, to demonstrate $R_{\infty^-} = 1 - Q_{0^+} = 1$, a verification of $Q_{0^+} = 0$ is required. If we would not have $Q_{0^+} = 0$, an eigenelement $h = Q_0 + h$ of Q_{0^+} would exist and, according to the second eigenvalue inequality with $\beta_1^* = 0^+$, it would be necessary that $(hHh) \leq 0$ from which, however, $h = 0$ would be obtained.

Since $R(\gamma_1^{\pm}, \gamma_2^{\pm}) = Q(\beta_2^{\mp}, \beta_1^{\mp})$, the eigenvalue inequalities of theorem 11 are transformed directly into those of theorem 12. Similarly, the G-symmetry of $R_{\gamma}\cdot$ in $\mathfrak{G}$ follows directly from this; however, the symmetry with respect to H follows only in $\mathfrak{G}$; anyhow, the $R_{\gamma}\cdot$ are defined by the above definition only in $\mathfrak{G}$. However, from the H-symmetry of $R_{\gamma}\cdot$ in $\mathfrak{G}$ there follows the H-boundedness in $\mathfrak{G}$; consequently, the $R_{\gamma}\cdot$ can be continued over the entire $\mathfrak{H}$ and, as is quite obvious, without losing their character as a spectral family.

This demonstrates the existence of the spectral family of $\mathfrak{G}$. The uniqueness is obtained from the fact that, inversely, the unique spectral family of H in the space $\mathfrak{G}$ can be obtained from that of G.

To prove the auxiliary theorem 12.2, we have to demonstrate that, for f from $\mathfrak{F}_1$, also $R_\gamma \cdot f$ is located in $\mathfrak{F}_1$. However, for all f, f_1 from $\mathfrak{F}_1$, we have /485

$$R_\gamma \cdot f\, H G f_1 = R_\gamma \cdot f\, G f_1 = f\, G R_\gamma \cdot f_1 = G f\, H R_\gamma \cdot f_1 = R_\gamma \cdot G f\, H f_1.$$

Since G in $\mathfrak{F}_1$ is selfadjoint, it follows that $R_\gamma \cdot f$ lies in $\mathfrak{F}_1$ and that $GR_\gamma \cdot f = R_\gamma \cdot Gf$.

For proving the auxiliary theorem 12.3, it should be noted that, according to the auxiliary theorem 12.2, the operator $GR_{\Delta\gamma}$ ($\Delta\gamma$, finite) is applicable in $\mathfrak{F}_1$. However, from the eigenvalue inequalities it follows that $GR_{\Delta\gamma}$ is symmetric with respect to H and is bounded, since this is the case for the pertinent form

$$f\, H G R_{\Delta\gamma} f = f\, G R_{\Delta\gamma} f$$

Consequently, the operator $GR\Delta\gamma$ can be continued over all of $\mathfrak{H}$; in this case, the operator will be denoted by $(GR\Delta\gamma)$ in $\mathfrak{F}_1$. Now, using f from $\mathfrak{F}_1$, we have

$$(G R_{\Delta\gamma})\, h\, H f = R_{\Delta\gamma}\, h\, H G f$$

which is valid for all h from $\mathfrak{F}$ and thus also for all h from $\mathfrak{H}$. Since G is selfadjoint, it follows that $R_{\Delta\gamma} h$ is located in $\mathfrak{F}_1$ and that $GR_{\Delta\gamma} h = (GR_{\Delta\gamma})h$ applies.

6. Complete Continuity and Discrete Spectrum

Hilbert and Weyl developed simple criteria for the fact that the spectrum of a given operator in a given interval consists only of discrete point eigenvalues of finite multiplicity. For these criteria, we will give a simple proof arrangement which simultaneously permits its extrapolation to unbounded operators. Let us first state the following:

A spectral family P, in a closed interval $\Delta\alpha = (\alpha_1-, \alpha_2+)$, has a discrete spectrum* if the eigenspace of $P_{\Delta\alpha}$ has a finite dimension.

It follows readily from this that $P_{\alpha'}$, within this interval, is constant in α' to within finitely many discontinuity points (point eigenvalues) whose eigenspaces have finite dimensions**.

In addition: The spectrum is denoted as discrete in a given interval if it is discrete in each closed subinterval. In general, let the spectrum be designated as discrete if it is discrete in each interval.

Let us now introduce the concept of complete continuity with respect to /486 a positive-definite form, as a generalization of the Hilbert concept of complete continuity. In a manner differing from that used by Hilbert, complete continuity can be characterized also by a property*** which is convenient in verifying the following theorems and is convenient to demonstrate in the application to differential operators.

Definition. Let gGg be a positive-definite form in a Hilbert space $\mathfrak{G} \leq \mathfrak{H}$. Then, let the form gVg in $\mathfrak{G}$ be completely continuous relative to G (denoted briefly as G-completely continuous) if a finite number of elements $h_1, h_2, \ldots, h_n$ from $\mathfrak{H}$ exist for each ϵ, so that we have

$$|g V g| \leq \sum_{\nu=1}^{n} (h_\nu H g)^2 + \epsilon (g G g).$$

This directly yields the following:

Lemma. Let a subspace $\mathfrak{Z}$ of $\mathfrak{G}$ have infinite dimensionality (i.e., have it develop infinitely many linearly independent elements). Then, $\mathfrak{Z}$ contains also

* Known also as "discrete point spectrum".

** The eigenvalues of a non-discrete point spectrum may lie densely or may have infinite multiplicity.

*** This is related with the characterization of complete continuity, preferred by Hellinger and Toeplitz in their article in this Encyclopedia.

an element $z \neq 0$, for which

$$|zVz| \leq \varepsilon(zGz)$$

is valid, i.e., also an element z with zGz = 1 and arbitrarily small zVz.

Since the space $\mathfrak{Z}$ has an infinite dimensionality, it definitely contains an element which is orthogonal on arbitrarily many $h_1, h_2, \ldots, h_n$.

The Hilbert criterion (Bibl.5.2) then reads:

Theorem 16. A form hVh in $\mathfrak{H}$, which is H-completely continuous, has a discrete spectrum in each interval not containing zero*.

Weyl's criterion (Bibl.12.3) refers to the modification of the spectrum of a bounded form if a completely continuous form is added. We will be satisfied here with extrapolating this criterion to positive-semibounded forms**.

Theorem 17. Let the form G in $\mathfrak{G} \leq \mathfrak{H}$ be semibounded with the lower bound $\underline{\gamma} > 0$ and let it be closed. Let the form gVg be completely continuous in the Hilbert space $\mathfrak{G}$ with the measure **G**.

Then, **G** + **V** has a discrete spectrum below $\underline{\gamma}$ (i.e., in each closed interval below $\underline{\gamma}$).

Auxiliary theorem 18. If the unitary form H is completely continuous with respect to G, then **G** itself has a discrete spectrum in any case.

The proofs for the mentioned theorems become entirely simple if, in /487 accordance with the method applied by F.Rellich (Bibl.8) in investigating the spectrum of differential equations, the above lemma is used as basis.

This lemma will be applied first in proving the first criterion (theorem 16). Assuming that an interval $\Delta\alpha = (\alpha_1^-, \alpha_2^+)$ not containing 0 would exist in

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* The inverse also applies.

** An extrapolation of the general Weyl criterion to arbitrary unbounded operators will be performed elsewhere.

which the spectrum of V is not discrete so that the eigenspace $\mathfrak{Z}$ of the spectral family of V, belonging to $\Delta\alpha$, would be of infinite dimensionality, i.e., would have many linearly independent eigenelements z, then the lemma would indicate that also a z exists in $\mathfrak{Z}$ for which the zHz would equal 1 and for which zVz would be arbitrarily small. However, because of $\alpha_1 > 0$ or $\alpha_2 < 0$, this contradicts the eigenvalue inequalities which require that

$$\alpha_1 (z\,H\,z) \leqq z\,V z \leqq \alpha_2 (z\,H\,z)$$

For proving the second criterion (theorem 17), we assume that, to an interval with the upper bound $\underline{\gamma}(1 - \epsilon) < \underline{\gamma}$, there would belong an eigenspace $\mathfrak{Z}$ of G + V with infinitely many linearly independent elements z. For these, the eigenvalue inequality

$$z\,G\,z + z\,V z \leqq \underline{\gamma}\,(1 - \varepsilon)\,(z\,H\,z);$$

exists; however, this inequality is in contradiction with $zGz \geq \underline{\gamma}(zHz)$ if, according to theorem 3, we select an element z from the subspace $\mathfrak{Z}$ of the Hilbert space $\mathfrak{G}$, for which zGz = 1 but for which zVz is arbitrarily small.

The auxiliary theorem 18 follows directly from the fact that, for the eigenelements z of $(\alpha_1^{\cdot}, \alpha_2^{\cdot})$, the second eigenvalue inequality

$$z\,G\,z \leqq \alpha_2\,(z\,H\,z)$$

exists; for an eigenelement z with zGz = 1 and arbitrarily small zHz, this leads to a contradiction.

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SPECTRAL THEORY OF SEMIBOUNDED OPERATORS AND THEIR USE IN SPECTRAL ANALYSIS OF DIFFERENTIAL OPERATORS. PART II*

K.Friedrichs

1. Spectral Theory of Differential Operators of the Second Order

In this second part of our paper, the theory of semibounded operators will be applied to linear differential operators of the second order, so as to obtain their spectral analysis. Here, we restrict ourselves to a few typical cases whose treatment will adequately demonstrate the generality of the method.

Let the operator in question be the "potential operator"

$$G = -\sum_{\nu=1}^{n} \frac{\partial^2}{\partial x_\nu^2} + v(x_1, \ldots, x_n);$$

The elements to which this operator is to be applied are functions f of the variables $x_1, \ldots, x_n$. The treated cases differ primarily by a different selection of the domain of the variables $x_1, \ldots, x_n$ and by different boundary conditions. To demonstrate that also quantum-theoretical-energy operators can be classified with Schrödinger's representation, we simultaneously treated the case that the "auxiliary potential" v has a singularity in one point. In addition, v (if necessary, by addition of a constant) has been selected so large that G becomes positive-semibounded.

Thus, we differentiate the following:

(1) Case of the infinite domain with regular v.

Here, the domain Γ is the entire $(x_1, \ldots, x_n)$ space. No special boundary condition need be established.

* Part I of this paper was published in Math. Annalen, Vol.109, pp.465-487.

(2) Case of the infinite domain with singular v; $n > 1$*.

Here, v may become infinite, in a still to be indicated manner, on approach to the point $x_1 = \ldots = x_n = 0$.

(3) Case of the finite spherical domain with regular v, at the boundary /686 condition

$$f = 0.$$

(4) Case of the finite spherical domain with regular v, at the boundary condition

$$\frac{\partial f}{\partial r} = 0.$$

Here, we have been satisfied in using, as finite domain, either a sphere, a circle, or a line segment since for overcoming the difficulties inherent in the nature of the domain our theory offers no new viewpoints. The same is true for the selection of the boundary conditions which, incidentally, will be somewhat weakened. Full treatment was given only to cases of the dimensionalities $n = 1, 2, 3$.

For the dimension $n = 1$, the entire theory can be represented in a much simpler manner; the cases of finite domain are accessible readily to conventional methods (for example, calculus of variations). We included these cases so as to clarify their coordination with the general theory.

2. Notations

Let a point of the variable space be denoted also by $x = (x_1, \ldots, x_n)$. We then pose

$$r = |x| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

* The general theory of semibounded differential operators with singularities at one variable, $n = 1$, will be presented elsewhere.

The basic domain Γ is to be as follows: in the case

(1) the total x-space,

(2) the total x-space without $x = 0$,

(3) (4) the "sphere" $r < R$.

We are using the following abbreviations (at $\rho \leq R$):

Ω_ρ for the spherical area $r = \rho$,

K_ρ for the sphere $r < \rho$,

$K_{\rho,P}$ for the spherical shell $\rho \leq r < P$.

In addition (at $\sigma > 0$) Γ_σ is to mean: in the case

(1) $\Gamma_\sigma = K_{1/\sigma}$,

(2) $\Gamma_\sigma = K_{\sigma,1/\sigma}$,

(3) (4) $\Gamma_\sigma = K_{R-\sigma}$,

so that, as $\sigma \to 0$, the domain Γ_σ exhausts the entire domain Γ.

As "square" about $x = \xi$ with the side 2δ, we denote the domain $|x_1 - \xi_1| < \delta$, ..., $|x_n - \xi_n| < \delta$ or, abbreviated, $[x - \xi] < \delta$. /687

The integration over the variable space will be denoted by

$$\int \ldots dx = \iint r^{n-1} \ldots dr\, d\omega$$

The improper integral over the total space $\int_\Gamma \ldots dx$ is to mean the limiting value of $\int_{\Gamma_\sigma} \ldots dx$ as $\sigma \to 0$.

3. Function Spaces and Operator

First, we will define the Hilbert space $\mathfrak{H}$, the space $\mathfrak{G}$ of the form G, and the space $\mathfrak{F}$ of the operator G. For this, dense subspaces known as "function spaces" will be given, formed by functions with simple differentiability properties; these will then be closed to spaces of ideal elements.

In order to form the Hilbert space $\mathfrak{H}$, we first will derive the subspaces

$\mathfrak{H}'$ and $\mathfrak{H}''$. As elements h, the subspaces $\mathfrak{H}''$ resp. $\mathfrak{H}'$ possess all functions $h(x)$, continuous or piecewise continuous* in Γ, of the variables

$$x = (x_1, \ldots, x_n),$$

for which

$$h\,H\,h = \int_{\Gamma} h^2(x)\,dx$$

exists. We will use H as the measure. It is known that $\mathfrak{H}'$ is separable** but not closed. According to theorem 2 (Part I), the subspace $\mathfrak{H}'$ can be uniquely continued to a closed Hilbert space $\mathfrak{H}$, specifically by adjunction of ideal elements h for which also the measure hHh is explained, so that $\mathfrak{H}'$ is dense in $\mathfrak{H}$. It is true that these ideal elements can be realized by quadratically $\mathfrak{L}$-integratable functions but there is no advantage in making use of this fact for /688 the case of $n > 1$.

We will use the following notation also for the ideal elements:

$$h\,H\,h = \int_{\Gamma} h^2\,dx;$$

If the elements h are functions $h(x)$, the variable x will be visualized in each case.

As subspaces of $\mathfrak{H}''$, let the spaces $\mathfrak{G}''$ and $\mathfrak{G}'$ be explained. Let $\mathfrak{G}''$ resp. $\mathfrak{G}'$ consist of all such functions $g(x)$ of $\mathfrak{H}''$ that have continuous or piecewise

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* A function $h(x)$ will be designated as piecewise continuous if it is not defined on a finite number of planes $x_\nu = \xi_\nu = \text{const}$, spherical surfaces $r = \rho = \text{const}$, or finite number of points $x = \xi$ and if it is otherwise continuous. The integrals $\int_\Gamma h^2\,dx$ are to mean limiting values for $\epsilon \to 0$ of the integrals over the domains formed from Γ by exclusion of the $|x_\nu - \xi_\nu| \le \epsilon$, $|r - \rho| \le \epsilon$, $|x - \xi| \le \epsilon$. If $\int_\Gamma h_1^2\,dx$, $\int_\Gamma h_2^2\,dx$ exists, then we also have $\int_\Gamma h_1 h_2\,dx$.

** The linear combinations of the elements $e_{\xi,\delta}$ of $\mathfrak{H}'$

$$e_{\xi,\delta} = \frac{1}{(2\delta)^n} \quad \text{in} \quad [x - \xi] < \delta, \qquad e_{\xi,\delta} = 0 \quad \text{external}$$

are known to lie H-dense in $\mathfrak{H}'$.

continuous first derivatives in Γ and for which the forms

$$g\,D\,g = \int_\Gamma \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 dx \qquad \left[\sum \left(\frac{\partial}{\partial x} g\right)^2 = \sum_{\nu=1}^{n} \left(\frac{\partial}{\partial x_\nu} g\right)^2\right],$$

$$g\,V\,g = \int_\Gamma v(x)\, g^2(x)\, dx$$

and thus - with $\mathbf{G} = \mathbf{D} + \mathbf{V}$ -

$$g\,G\,g = \int_\Gamma \left\{ \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 + v(x)\, g^2(x) \right\} dx$$

exist.

In the case (3) at a finite domain $\Gamma = K_R$, the functions $g(x)$ of $\mathfrak{G}'$ must already be subjected to the boundary condition of vanishing on Ω_R. This is replaced by the less strict condition:

$$\int_{\Omega_{R-\sigma}} g^2(x)\, dx \underset{\sigma \to 0}{\to} 0 \quad \text{in } \mathfrak{G}'. \tag{1}$$

In the case (4) as well as in the cases (1), (2), no boundary condition for $\mathfrak{G}'$ is to be stipulated.

Then, the following conditions are formulated for the auxiliary potential $v(x)$:

<u>Case (1), (3), (4)</u>.

Let $v(x)$ be continuous in Γ^* and be bounded downward by

$$v(x) \geqq 1. \tag{2}$$

<u>Case (2)</u>: $n > 1$.

Using any $\mathbf{P} > 0$, we assume that

1) For $n \geq 3$,

$$\varphi(r) = \frac{n-2}{2} \frac{1}{r} \qquad\qquad 0 < r \leqq \mathbf{P}.$$

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* In the case $n = 1$, a piecewise continuity is sufficient; in the untreated case $n > 3$, stricter requirements will have to be made on v.

2) For n = 2

$$\varphi(r) = \frac{1}{2}\frac{1}{\ln\frac{A}{r}}\frac{1}{r} \qquad 0 < r \leqq P$$

for any $A > P$.

Then, let $v(x)$ be continuous in Γ and let one v, one P, and a number Θ from $0 \leq \Theta < 1$ exist so that, at $\varphi(r) = 0$ for $r > P$, we have

$$v(x) \geqq \underline{v} - \Theta\,\varphi^2(r) \tag{2_Θ}$$

Of the constant $\underline{v}$ we also require that it be sufficiently large, namely,

$$v \geqq 1 + \Theta k,$$

where the constant $k > 0$ is to be determined in accordance with Appendix (2.2)*.

We will denote also the case (2) at $\Theta = 0$ by $(2)_0$, and at $\Theta \neq 0$ by $(2)_\Theta$.

With the form $G = D + V$, explained in $\mathfrak{G}'$, the following estimate exists:

$$g\,G\,g \geqq (1-\Theta)\,g\,D\,g + (g\,H\,g), \tag{3_Θ}$$

so that the form G is positive-semibounded. This fact, in the cases (1), $(2)_0$, (3), (4) with $\Theta = 0$ follows directly from $\underline{v} \geq 1$. In the case $(2)_\Theta$, we refer to the fact that, according to Appendix (2.2) and (2.3), the estimate

$$\int_{K_P} \left\{ \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 - \varphi^2(r)\,g^2(x) \right\} dx + k \int_{K_P} g^2(x)\,dx \geqq 0$$

is valid so that**

$$\int_{K_P} \left\{ \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 + v(x)\,g^2 \right\} dx \geqq (1-\Theta) \int_{K_P} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx + \int_{K_P} g^2\,dx. \tag{4}$$

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* To coordinate the Schrödinger problem, it is necessary to set $v(x) = -\frac{c}{r} +$ + const.

** Together with eq. $(3)_\Theta$, it follows from eq. (4) that

$$g\,G\,g \geqq \int_{\Gamma - K_P} v\,g^2\,dx.$$

which will be noted for later use.

The space $\mathfrak{G}'$ can then be closed, with **G** as measure, to a Hilbert space $\mathfrak{G}$ of elements g from $\mathfrak{H}$. According to theorem 3 (Part I), this follows from the fact (to be proved below) that the form G in a G-dense subspace $\mathfrak{F}'$ of $\mathfrak{G}''$, leads to an operator G. For the ideal elements from $\mathfrak{G}$, the forms **D**, **V**, and **G** will again be symbolically represented by the corresponding integrals. In that case, the inequality $(3)_\Theta$ will apply also in the space $\mathfrak{G}$.

The spaces $\mathfrak{F}''$ and $\mathfrak{F}'$ consist* of all functions f(x) from $\mathfrak{G}''$ that possess continuous or piecewise continuous second derivatives in Γ and for which the function $-\Delta f(x) + v(x)f(x)$ (for abbreviation, we are using $\Delta = \sum_{\nu=1}^{n} \frac{\partial^2}{\partial x_\nu^2}$) is located in $\mathfrak{H}''$ resp. in $\mathfrak{H}$. In $\mathfrak{F}'$, the operator G is explained by /690

$$G = -\Delta + v.$$

In the case (4) of a finite domain $\Gamma = K_\rho$, the boundary condition $\frac{\partial}{\partial r} f(x) = 0$ on Ω_R must be stipulated. We establish this condition in the weakened form: With each function g(x) of $\mathfrak{G}''$, let the following be valid for f(x) from $\mathfrak{F}'$ and $\mathfrak{F}''$ in the case (4):

$$\int_{\Omega_\varrho} g(x) \frac{\partial}{\partial r} f(x)\, d\omega \underset{\varrho \to R}{\to} 0. \tag{5}$$

In fact, $\mathfrak{F}'$ is G-dense located in $\mathfrak{G}''$ and the operator G in $\mathfrak{F}'$ belongs** to the form G since, for each f from $\mathfrak{F}'$ and for each g from $\mathfrak{G}'$ "Green's transformation"

$$f\,G\,g = G\,f\,H\,g; \tag{6}$$

is valid. Its proof will be given in Appendix 3.

* In the case (1) it is sufficient to stipulate, instead of the correlation with $\mathfrak{G}''$, the existence of continuous first derivatives, from which the existence of the integral **G** follows automatically; see Appendix 3, auxiliary theorem.

** By twice continuously differentiable functions from $\dot{\mathfrak{G}}'$ (see Sect. 5.2), each function from $\mathfrak{F}''$ can be G-approximated. Then, $\dot{\mathfrak{G}}'$ is G-dense in $\mathfrak{G}'$.

The operator G in $\mathfrak{F}'$, according to theorem 4 (Part I), can be continued uniquely to a closed operator G in a space $\mathfrak{F}$ of elements f. The space $\mathfrak{F}$ is located in $\mathfrak{G}$ [note on theorems 3 and 4 (Part I)], and eq.(6) remains valid in $\mathfrak{F}$ and $\mathfrak{G}$.

4. Spectral Analysis; Theorems

Naturally, it cannot be expected that the originally explained operator G in $\mathfrak{F}'$ possesses a spectral resolution since it is not closed. However, this can be stipulated for the uniquely coordinated closed operator G in $\mathfrak{F}$. Beyond this, it is established that the eigenelements of finite intervals* already lie in the space $\mathfrak{F}''$ and thus are twice continuously differentiable and permit successive application of G as often as desired.

The spectral resolvability of the operator G in $\mathfrak{F}$ is due to its self- /691 adjointness. To prove that G is selfadjoint in $\mathfrak{F}$ is one of our main tasks. For this, we start from the premise that the operator G in $\mathfrak{F}'$, which had been closed to G in $\mathfrak{F}$, can be continued in a different manner to a selfadjoint operator. Namely: The form G in $\mathfrak{F}'$, belonging to G in $\mathfrak{F}'$, is formed first and then closed to **G** in $\mathfrak{G}$ according to theorem 3 (I), after which the selfadjoint operator G, belonging to **G** in $\mathfrak{G}$ and explained in $\mathfrak{F}_1 \leq$ G is formed according to theorem 7 (I)**. Then, G in $\mathfrak{F}_1$ is a continuation of G in $\mathfrak{F}'$ and thus also of G in $\mathfrak{F}$ [auxiliary to theorem 7 (I)].

Consequently, it is sufficient to demonstrate $\mathfrak{F} = \mathfrak{F}_1$. For this, we refer to the criteria established earlier for theorem 10 (I). Here, $(\mathfrak{F}_n)$:$\mathfrak{F}_n$ lies

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* This means the eigenelements of the difference projection operators $R_{\Delta\gamma}$ of the spectral family R_γ. of G, belonging to finite intervals $\Delta\gamma$.

** Equation (6) applies also for g in $\mathfrak{G}$, f in $\mathfrak{F}_1$.

in $\mathfrak{F}$ or else $(\mathfrak{G}_n)$: $\mathfrak{G}_n$ lies in $\mathfrak{F}$. The elements f from $\mathfrak{F}_1$, for which Gf lies in

$$\mathfrak{G}, \mathfrak{F}_1, \mathfrak{G}_1, \mathfrak{F}_2, \dots$$

formed the iterated spaces

$$\mathfrak{G}_1, \mathfrak{F}_2, \mathfrak{G}_2, \mathfrak{F}_3, \dots$$

The first of our theorems will be differently formulated, depending on the number of variables involved.

A. <u>One variable</u>: $n = 1$.

Theorem 1A:

1. $\mathfrak{G}$ lies in $\mathfrak{H}''$,
2. $\mathfrak{F}_1$ lies in $\mathfrak{G}''$,
3. $\mathfrak{G}_1$ lies in $\mathfrak{F}''$.

B. <u>Two and three variables</u>: $n = 2, 3$.

Theorem 1B:

1. $\mathfrak{F}_1$ lies in $\mathfrak{H}''$,
2. $\mathfrak{G}_1$ lies in $\mathfrak{G}''$,
3. $\mathfrak{F}_2$ lies in $\mathfrak{F}''$.

From theorem 1, it then follows that:

<u>Theorem 2</u>:

$$\mathfrak{F} = \mathfrak{F}_1,$$

i.e., the operator G in $\mathfrak{F}$ is selfadjoint.

In fact, the criterion $(\mathfrak{G}_1)$ is satisfied in the case A while the criterion $(\mathfrak{F}_2)$ is satisfied in the case B.

Furthermore, it follows directly from theorems 1 and 2 that:

<u>Theorem 3</u>. If the operator G is arbitrarily often applicable to f from $\mathfrak{F} = \mathfrak{F}_1$, then f, Gf, $G^2 f$, ..., $G^n f$, ... will lie in $\mathfrak{F}''$.

Theorem 4. A spectral family R_γ, in the sense of theorem 12 (I), exists for the form G in $\mathfrak{G}$ and for the operator G in $\mathfrak{F}$.

2) The eigenelements of the difference operators $R_{\Delta\gamma}$ of finite intervals /692 $\Delta\gamma = (\gamma_1, \gamma_2)$ are functions $f_{\Delta\gamma}(x)$ from $\mathfrak{F}''$. With these, the following eigenvalue inequalities apply:

$$\gamma_1 \int_\Gamma f_{\Delta\gamma}^2(x)\, dx \leqq \int_\Gamma \left\{ \sum \left(\frac{\partial}{\partial x} f_{\Delta\gamma}(x)\right)^2 + v(x)\, f_{\Delta\gamma}^2(x) \right\} dx \leqq \gamma_2 \int_\Gamma f_{\Delta\gamma}^2(x)\, dx.$$

Proof. Theorem 4.1, according to theorem 2, follows from theorem 12 (I). Then, according to the auxiliary theorem 12.3 (I), theorem 4.2 follows from theorem 3.

5. Preparations for the Proof of Theorem 1

5.1 Mean Values

A principal aid is constituted by the mean-square values of elements h from $\mathfrak{H}$ over squares $[x - \xi] < \delta$ from Γ. For this, we introduce the elements $e_{\xi,\delta}$ from $\mathfrak{H}'$, which are defined by

$$e_{\xi,\delta}(x) = \frac{1}{(2\delta)^n} \quad \text{in} \quad [x-\xi] < \delta,$$
$$e_{\xi,\delta}(x) = 0 \quad \text{out of} \quad [x-\xi] < \delta.$$

The mean-square values of h from $\mathfrak{H}$ are then explained by

$$e_{\xi,\delta}\, H\, h = \int_\Gamma e_{\xi,\delta}\, h\, dx,$$

for which we also use the notation

$$\frac{1}{(2\delta)^n} \int_{[x-\xi]<\delta} h\, dx.$$

The limiting values of the mean-square values, on contraction of the square to the center, will be denoted - so far as they exist - by h_ξ:

$$\frac{1}{(2\delta)^n} \int\limits_{|x-\xi|<\delta} h\,dx \underset{\delta\to 0}{\to} h_\xi$$

5.2 Spaces $\dot{\mathfrak{H}}'$, ..., $\dot{\mathfrak{F}}''$

Frequently, functions of the spaces $\mathfrak{H}'$, ..., $\mathfrak{F}''$ will occur which vanish outside of any domain Γ_σ. The subspaces, formed by these functions and obviously lying densely in $\mathfrak{H}$, are denoted respectively by

$$\dot{\mathfrak{H}}', \ldots, \dot{\mathfrak{F}}''.$$

For the functions h(x) from $\dot{\mathfrak{H}}'$, the operator V is explained by

$$Vh = v(x)\,h(x)\,.$$

For elements f from $\dot{\mathfrak{F}}'$, the operator

$$\Delta f = \sum_{\nu=1}^{n} \frac{\partial^2}{\partial x_\nu^2} f(x)$$

is explained, so that for f from $\dot{\mathfrak{F}}'$ /693

$$\int\limits_{\Gamma} \sum \left(\frac{\partial}{\partial x} g \frac{\partial}{\partial x} f\right) dx = -\int\limits_{\Gamma} g\,\Delta f\,dx \tag{7}$$

applies directly if g belongs to $\dot{\mathfrak{G}}'$. This transformation can be extrapolated to all g from $\mathfrak{G}$ since the form $\int\limits_{\Gamma} \Sigma \left(\frac{\partial}{\partial x} g\right)^2 dx$ is boundedly explained in $\mathfrak{G}$.

5.3 The Operator S_σ

Another aid is an operation which, from each element, produces another element which vanishes at a sufficient distance in the neighborhood of the singular point. For this, we select an arbitrarily often differentiable function $s_\sigma(x)$ with the following properties:

$$s_\sigma(x) = 1 \quad \text{in} \quad \Gamma_\sigma,$$

$$0 < s_\sigma(x) \leqq 1 \quad \text{in} \quad \Gamma_{\sigma/2} - \Gamma_\sigma,$$
$$s_\sigma(x) = 0 \quad \text{out of} \quad \Gamma_{\sigma/2}. \tag{8}$$

The operator S_σ in $\mathfrak{H}'$ is explained by

$$S_\sigma h = s_\sigma(x)\, h(x);$$

Because of its boundedness, it must be continued over the entire $\mathfrak{H}$. Obviously, S_σ will produce other elements from $\mathfrak{G}'$ and $\mathfrak{G}''$. We can demonstrate that S_σ is bounded in $\mathfrak{G}$, with G as measure. For this purpose, it is obviously sufficient to demonstrate the boundedness in $\mathfrak{G}'$, i.e., that, in this condition – note* that S_σ is not symmetric with respect to G – the following is valid for two elements g and g_1 of $\mathfrak{G}'$:

$$|S_\sigma g\, G\, g_1|^2 \leqq C\,(g\, G\, g)\,(g_1\, G\, g_1)$$

If $\left|\frac{\partial s_\sigma}{\partial x}\right| \leq c_1$, $|v(x)| \leq c_2$ in $\Gamma_{\sigma/2}$, then we have

$$|S_\sigma g\, D\, g_1| = \left|\int\limits_\Gamma s_\sigma(x) \sum \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g_1(x)\, dx\right.$$
$$\left. + \int \sum \left(\frac{\partial}{\partial x} s_\sigma(x) \frac{\partial}{\partial x} g_1(x)\right) g(x)\, dx\right| \leqq \sqrt{g\, D\, g}\sqrt{g_1\, D\, g_1} + c_1\sqrt{g\, H\, g}\sqrt{g_1\, D\, g_1},$$
$$|S_\sigma g\, V\, g_1| \leqq c_2\sqrt{g\, H\, g}\sqrt{g_1\, H\, g_1}.$$

Taking the inequality $(3)_{\mathfrak{G}}$ into consideration and using suitable constants C, we will actually obtain $|S_\sigma g G g_1| \leq C\sqrt{gGg}\sqrt{g_1 G g_1}$.

6. Three Lemmas /694

Lemma 1. If, for the element h from $\mathfrak{H}$, the limiting value h_ξ exists continuously in ξ in such a manner that $\int\limits_\Gamma e_{\xi,\delta} h dx \to h_{\xi_0}$, for $\xi = \xi_\delta \to \xi_0$, then it follows that:

1) the integral

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* See previous footnotes on pp.8 and 10 of Part I.

$$\int_{\Gamma} h_x^2\, dx,$$

exists.

2) h is a function h(x) from $\mathfrak{H}''$ with $h(x) = h_x$.

<u>Proof</u>. Let us first note the relation valid for continuous h_x:

$$\frac{1}{(2\delta)^n} \int\limits_{[x-\xi]<\delta} h_x\, dx = \frac{1}{(2\delta)^n} \int\limits_{[x-\xi]<\delta} h\, dx, \tag{9}$$

where the integration of the left-hand side is to be understood in the conventional sense. In fact, in all other cases we assume that $[x - \xi_1] < \delta_1$ is a square for which

$$\frac{1}{(2\delta_1)^n} \left| \int\limits_{[x-\xi_1]<\delta_1} h_x\, dx - \int\limits_{[x-\xi_1]<\delta_1} h\, dx \right| = \alpha > 0$$

is valid. Then, we resolve $[x - \xi_1] < \delta_1$ into 2^n subdomains of a side $2\delta_2 = \delta_1$. For one of these

$$[x - \xi_2] < \delta_2 = \frac{\delta_1}{2},$$

the following must then be valid:

$$\frac{1}{(2\delta_2)^n} \left| \int\limits_{[x-\xi_2]<\delta_2} h_x\, dx - \int\limits_{[x-\xi_2]<\delta_2} h\, dx \right| \geq \alpha.$$

Closing furthermore in this manner, a nesting sequence of domains

$$[x - \xi_\nu] < \delta_\nu = \frac{\delta_1}{2^\nu},$$

will be created, which converge toward a limiting point ξ_0. However, because of the continuity of h_ξ, we have for $\nu \to \infty$,

$$\frac{1}{(2\delta_\nu)^n} \int\limits_{[x-\xi_\nu]<\delta_\nu} h_x\, dx \to h_{\xi_0}, \text{ besides } \frac{1}{(2\delta_\nu)^n} \int\limits_{[x-\xi_\nu]<\delta_\nu} h\, dx \to h_{\xi_0},$$

from which the contradiction $|h_{\xi_0} - h_{\xi_0}| \geq \alpha$ originates.

It will be noted that the function $s_\sigma(x) h_x$ is an element

$$h_\sigma^* = s_\sigma(x)\,h_x$$

from $\mathfrak{H}''$. The above equation (9) is then, for all squares $[x - \xi] < \delta$ from Γ_τ, equivalent to

$$\int_\Gamma (h_\tau^* - h)\, e_{\xi,\delta}\, dx = 0.$$

If, now, $k = k(x)$ is an element from $\mathfrak{H}''$, then the function $S_\sigma k = s_\sigma(x)k(x)$ /695 can be approximated, by linear combinations, to the function $e_{\xi,\delta}$ for which $[x - \xi] < \delta$ is located in Γ_τ with $\tau = \frac{\sigma}{3}$, so that we will have

$$\int_\Gamma (h_\tau^* - h)\, S_\sigma k\, dx = 0.$$

Considering next $S_\sigma h_\tau^* = h_\sigma^*$, it follows that

$$\int_\Gamma (h_\sigma^* - S_\sigma h)\, k\, dx = 0.$$

Consequently, we have

$$S_\sigma h = h_\sigma^* = s_\sigma(x)\, h_x.$$

Then, for each element h from $\mathfrak{H}$, the following inequality applies:

$$\int_\Gamma (S_\sigma h)^2\, dx \leqq \int_\Gamma h^2\, dx,$$

since it is valid for the h from $\mathfrak{H}'$. Hence, we specifically have

$$\int_{\Gamma_\sigma} h_x^2\, dx = \int_{\Gamma_\sigma} h_\sigma^{*2}(x)\, dx \leqq \int_\Gamma h_\sigma^{*2}(x)\, dx = \int_\Gamma (S_\sigma h)^2\, dx \leqq \int_\Gamma h^2\, dx.$$

From this follows the first part of the statement, namely, the existence of the integral $\int_\Gamma h_x^2 dx$. This definitely establishes that h_x is a function $h^*(x)$ from $\mathfrak{H}''$ for which $S_\sigma h^* = s_\sigma(x)h_x = S_\sigma h$ so that

$$\int_\Gamma S_\sigma k\,(h^* - h)\, dx = 0$$

is valid; however, since the $S_\sigma k$ are dense in $\mathfrak{H}$, it follows that

$$h^* = h.$$

This constitutes the second part of the statement.

In addition, we have:

Lemma 2. Let an element g from $\mathfrak{G}$ be a function g(x) from $\mathfrak{H}''$ and let g(x) be continuously differentiable.

Then,

1) the relations

$$\int_{\Gamma} \left\{ \sum \left(\frac{\partial}{\partial x} g(x) \right)^2 \right\} dx \quad \text{and} \quad \int_{\Gamma} v(x) g^2(x)\, dx,$$

exist;

2) the element g is located in $\mathfrak{G}''$.

For the proof, we will use the form G_σ which is assumed to be explained in $\mathfrak{G}'$ by

$$g\, G_\sigma\, g = \int_{\Gamma_\sigma} \left\{ \sum \left(\frac{\partial}{\partial x} g(x) \right)^2 + v(x) g^2(x) \right\} dx - \Theta\, \psi(\sigma) \int_{\Omega_\sigma} g^2(x)\, d\omega$$

with $\psi(r) = \varphi(r) r^{n-1}$ and $\Theta = 0$, except in the case $(2)_\Theta$.

We have /696

$$g\, G_\sigma\, g = (1-\Theta) \int_{\Gamma_\sigma} \sum \left(\frac{\partial}{\partial x} g \right)^2 dx + \int_{\Gamma_\sigma} \{ v + \Theta\, \varphi^2(r) \} g^2\, dx$$
$$+ \Theta \left[\int_{\Gamma_\sigma} \left\{ \sum \left(\frac{\partial}{\partial x} g \right)^2 - \varphi^2(r)\, g^2 \right\} dx - \psi(\sigma) \int_{\Omega_\sigma} g^2\, d\omega \right].$$

From the properties of v(x) - and in the case $(2)_\Theta$ according to Appendix (2.1), (2.3) - it is obvious that $gG_\sigma g$ does not decrease as $\sigma \to 0$ and actually tends to gGg. Specifically, we have

$$g\, G_\sigma\, g \leq g\, G\, g.$$

Thus, the form G_σ is bounded in $\mathfrak{G}'$ and therefore can be continued over the entire $\mathfrak{G}$, in which case this inequality remains applicable.

Since the operator S_σ in $\mathfrak{G}'$, with G as measure, is bounded, the identity

$$S_\sigma g\, G_\sigma S_\sigma g = g\, G_\sigma g$$

valid in $\mathfrak{G}'$ is also valid in $\mathfrak{G}$.

If the continuously differentiable function g(x) lies in $\mathfrak{G}$ and $\mathfrak{H}''$, then it is definite that $S_\sigma g = s_\sigma(x)g(x)$ lies in $\mathfrak{G}''$ and that, for $\sigma < \sigma_0$ – in the case $(2)_\Theta$ according to Appendix (2.0) – we obtain

$$\begin{aligned}
&(1-\Theta)\int_{\Gamma_\sigma} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx + \int_{\Gamma_\sigma} (v + \Theta\, \varphi^2(r))\, g^2\, dx \\
\leqq &\int_{\Gamma_\sigma} \left\{ \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 + v(x)\, g^2(x)\right\} dx - \Theta\, \psi(\sigma) \int_{\Omega_\sigma} g^2(x)\, d\omega \\
= &\, S_\sigma g\, G S_\sigma g = g\, G_\sigma g \leqq g\, G g.
\end{aligned}$$

Consequently, the left-hand side remains bounded with increasing σ. From this, we first obtain the existence of the integral

$$\int_{\Gamma} \sum \left(\frac{\partial}{\partial x} g(x)\right)^2 dx$$

and – in the case $(2)_\Theta$ according to lemma (2.3) – the existence of

$$\int_{\Gamma} v(x)\, g^2(x)\, dx.$$

This proves the first statement of lemma 2. The second portion, namely, that g lies in $\mathfrak{G}''$, follows directly for the cases (1), (2), (4) from the explanation of $\mathfrak{G}''$.

In the case (3), we still have to prove that g(x) also satisfies the boundary condition

$$\int_{\Omega_{R-\sigma}} g^2(x)\, dx \underset{\sigma \to 0}{\to} 0 \qquad (1)$$

For this, we will demonstrate that this boundary condition is equivalent to the following: /697

A constant $C > 0$ exists so that, for $\rho > \rho_0 > 0$, the following is valid:

$$\int_{\Omega_\varrho} g^2\,d\omega \leqq 2\,C\,(R-\varrho) \int_{K_{\varrho,R}} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx. \qquad (1^*)$$

In fact, from eq.(1^*), the validity of eq.(1) follows directly. Conversely, if eq.(1) is satisfied, the inequality

$$\int_{\Omega_\varrho} g^2\,d\omega \leqq 2 \int_{\Omega_\tau} g^2\,d\omega + 2\,C\,(R-\varrho) \int_{K_{\varrho,R}} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx, \quad \varrho < \tau < R$$

is used, as it is obtained from eq.(1.1) in the Appendix, with $C = \rho_0^{1-n}$. If

$$\tau = R - \sigma \to R$$

we will obtain eq.(1^*).

The inequality (1^*) is directly found as equivalent to the following condition:

A constant C exists such that, for all ρ_1, ρ_2 from

$$\varrho_0 \leqq \varrho_1 < \varrho_2 < R$$

the following is valid:

$$\int_{K_{\varrho_1,\varrho_2}} g^2\,dx \leqq 2\,C\,(R-\varrho) \int_{\varrho_1}^{\varrho_2} r^{n-1}\,dr \int_{K_{\varrho_1,R}} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx. \qquad (1^{**})$$

This is so since the inequality (1^*) follows from this for $\rho_2 \to \rho_1 = \rho$. Conversely, the same relation proves the existence of eq.(1^{**}) taking into consideration that an intermediary value ρ exists, so that we have

$$\int_{K_{\varrho_1,\varrho_2}} g^2\,dx = \int_{\varrho_1}^{\varrho_2} r^{n-1}\,dr \int_{\Omega_\varrho} g^2\,d\omega.$$

Both sides of eq.(1^{**}) represent forms bounded in $\mathfrak{G}'$, which must be continued on $\mathfrak{G}$. Consequently, eq.(1^{**}) is applicable also to the function g(x) from $\mathfrak{H}''$ and $\mathfrak{G}$ for which we just have proved the existence of the integral $\int_\Gamma \Sigma \left(\frac{\partial}{\partial x} g\right)^2 dx$. For g(x), these continued forms are represented also by the integrals. Consequently, the condition (1) is satisfied for these functions

$g(x)$, which means that they are located in $\mathfrak{G}''$.

Lemma 3. Let an element f from $\mathfrak{F}_1$ be a function from $\mathfrak{G}''$ and let $Gf = h(x)$ lie in $\mathfrak{H}''$ and let $f(x)$ have continuous second derivatives. Then, we have

1) $Gf = -\Delta f(x) + v(x)f(x)$;

2) f is located in $\mathfrak{F}''$.

Proof. Let $g(x)$ be an element from $\dot{\mathfrak{G}}''$. Then, since g is located in $\mathfrak{G}$ /698 and f in $\mathfrak{F}_1$, eq.(6) is applicable. Since g and f are located in $\mathfrak{G}''$ and Gf in $\mathfrak{H}''$, this equation is transformed into

$$\int_\Gamma g(x)\, h(x)\, dx = \int_\Gamma \left\{ \sum \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} f(x) + v(x)\, g(x)\, f(x) \right\} dx$$

or, if a partial integration is performed, into

$$\int_\Gamma g(x)\, h(x)\, dx = \int_\Gamma g(x) \{-\Delta f(x) + v(x) f(x)\}\, dx.$$

Since $\dot{\mathfrak{G}}''$ lies densely in $\mathfrak{H}$, it follows that

$$Gf = h(x) = -\Delta f(x) + v(x) f(x)$$

and $h(x)$ lying in $\mathfrak{H}''$ is transformed into f lying in $\mathfrak{F}''$, according to the explanation of this space.

Then, it only remains to demonstrate for the case (4) that $f(x)$ satisfies the boundary condition (5). The relation

$$\int_\Gamma \left\{ g(x)\, G f(x) - \sum \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} f(x) - v(x)\, g(x)\, f(x) \right\} dx = 0,$$

which is valid for all elements $g(x)$ from $\mathfrak{G}'$ states that this integral, extending over $\dot{K}_{R-\sigma}$, vanishes as $\sigma \to 0$. After conventional transformation, this will yield

$$\int_{\Omega_{R-\sigma}} g(x) \frac{\partial}{\partial x} f(x)\, dx \underset{\sigma \to 0}{\longrightarrow} 0 \tag{5}$$

for all functions $g(x)$ from $\mathfrak{G}'$.

7. Proof of Theorem 1A

The continuation of our proof, depending on the number n of independent variables, will be separately performed.

A. One Independent Variable, $n = 1$

We are making use of the basic solution

$$K(x-\xi) = -\tfrac{1}{2}|x-\xi|$$

of the operator $\Delta = \frac{d^2}{dx^2}$ and form

$$k_\xi = k_\xi(x) = -\tfrac{1}{2}s_\sigma(x)|x-\xi|,$$

where $s_\sigma(x)$ is to be selected in accordance with Section 5.3. Here, $k_\xi(x)$, as a function of x, is an element of $\mathfrak{G}'$ but not of $\mathfrak{F}'$ since the first derivative $\frac{d}{dx} k_\xi(x)$ is discontinuous at $x = \xi$. The second derivative $\frac{d^2}{dx^2} k_\xi(x) =$ /699
$= \Delta k_\xi(x)$, however, is again an element of $\mathfrak{H}''$ and vanishes in Γ_σ and external to $\Gamma_{\sigma/2}$.

Lemma 4A. For all g from $\mathfrak{G}$ resp. h from $\mathfrak{H}$,

$\int_\Gamma k_\xi h\,dx$	is continuously differentiable
$\int_\Gamma \Delta k_\xi h\,dx$	is continuous
$\int_\Gamma \frac{d}{dx}k_\xi \frac{d}{dx} g\,dx$	is continuous*

in ξ, if ξ lies in Γ_σ.

Proof. These properties can be assumed as known provided that the elements h, g are continuous or continuously differentiable functions. However, in this case they follow generally if it can be demonstrated that, for the three functions

$$h_\varepsilon = \frac{1}{\varepsilon}(k_{\xi+\varepsilon} - k_\xi), \quad h_\varepsilon = \frac{d}{d\xi}k_{\xi+\varepsilon}, \quad h_\varepsilon = \Delta k_{\xi+\varepsilon}$$

* See Section 5.2.

and for $g_\varepsilon = k_{\xi+\varepsilon}$, the forms

$$\int_\Gamma h_\varepsilon^2\, dx \quad \text{and} \quad \int_\Gamma \left(\frac{d}{dx} g_\varepsilon\right)^2 dx$$

remain bounded in ε. This boundedness is directly determinable according to the definition of $k_\xi(x)$.

Then, the ξ integrals

$$k_{\xi,\delta} = \frac{1}{2\delta} \int_{\xi-\delta}^{\xi+\delta} k_\eta\, d\eta.$$

are formed for the functions $k_\xi(x)$. Here, it will always be assumed that $|x - \xi| < \delta$ lies in Γ_σ. We then have:

Lemma 5A. The functions $k_{\xi,\delta}(x)$ are once continuously differentiable and piecewise twice continuously differentiable, so that

$$-\frac{d^2}{dx^2} k_{\xi,\delta}(x) = -\Delta k_{\xi,\delta}(x) = e_{\xi,\delta}(x) \quad \text{in} \quad \Gamma_\sigma.$$

Hence,

$$k_{\xi,\delta}(x) \text{ lies in } \dot{\mathfrak{H}}'.$$

Proof. We can calculate

$$\begin{aligned} k_{\xi,\delta} &= -\frac{1}{4\delta}\{(x-\xi)^2 + \delta^2\} \quad \text{in} \quad |x-\xi| < \delta \\ &= -\tfrac{1}{2}|x-\xi|, \quad \text{if x otherwise lies in } \Gamma_\sigma. \end{aligned}$$

External to Γ_σ, the quantity $k_{\xi,\delta}$ is arbitrarily often continuously differenti- /700 able to x and outside of $\Gamma_{\sigma/2}$, we have $k_{\xi,\delta} = 0$. From this, the statement is derived.

Lemma 6A. As $\delta \to 0$ and $\xi = \xi_\delta \to \xi_0$, for each g from $\mathfrak{G}$, we have

$$\int_\Gamma \frac{d}{dx} k_{\xi,\delta} \frac{d}{dx} g\, dx \to \int_\Gamma \frac{d}{dx} k_{\xi_0} \frac{d}{dx} g\, dx$$

$$\int_\Gamma (\Delta k_{\xi,\delta} + e_{\xi,\delta})\, g\, dx \to \int_\Gamma \Delta k_\xi g\, dx.$$

Proof. The first statement follows from the convergence

$$\frac{d}{dx} k_{\xi,\delta} \to \frac{d}{dx} k_{\xi_0}$$

uniform in x except in a vicinity $|x - \xi_0| < \epsilon$ of $x = \xi_0$. However,

$$\int\limits_{|x-\xi_0|<\epsilon} \left(\frac{d}{dx} k_{\xi_0,\delta}\right)^2 dx \leqq \frac{\epsilon}{2} \underset{\epsilon \to 0.}{\to} 0.$$

The second statement follows from the uniform convergence

$$\Delta k_{\xi,\delta} + e_{\xi,\delta} = \Delta k'_{\xi,\delta} \to \Delta k_\xi \text{ outside of } \Gamma_\sigma,$$
$$\Delta k_{\xi,\delta} + e_{\xi,\delta} = 0 = \Delta k_\xi \text{ in } \Gamma_\sigma.$$

Now, we are in a position to prove the following theorem:

Theorem 1.1A:

$\mathfrak{G}$ lies in $\mathfrak{H}''$.

Let g be any element from $\mathfrak{G}$. Then, according to lemma 5A and because of the fact that $k_{\xi,\delta}$ lies in $\mathfrak{F}'$, we obtain in accordance with Section 5.2

$$\frac{1}{2\delta} \int\limits_{|x-\xi|<\delta} g\,dx = \int\limits_{\Gamma} \{e_{\xi,\delta} + \Delta k_{\xi,\delta}\}\, g\,dx + \int\limits_{\Gamma} \frac{d}{dx} k_{\xi,\delta} \frac{d}{dx} g\,dx;$$

Consequently, according to lemma 6A, a limiting value g_{ξ_0} exists as $\delta \to 0$ and $\xi = \xi_\delta \to \xi_0$, resulting in

$$g_\xi = \int\limits_{\Gamma} \Delta k_\xi\, g\,dx + \int\limits_{\Gamma} \frac{d}{dx} k_\xi \frac{d}{dx} g\,dx. \tag{11A}$$

According to lemma 4A, g_ξ is continuous in Γ_σ and, since σ had been arbitrary, also in Γ. Then, according to lemma 1, g is located in $\mathfrak{H}''$ with $g(x) = g_x$.

Theorem 1.2A:

$\mathfrak{F}_1$ lies in $\mathfrak{G}''$.

Proof. Let f be an element of $\mathfrak{F}_1 < \mathfrak{G}$ and let $Gf = h$ lie in $\mathfrak{H}$. According to theorem 1.1, $f = f(x)$ lies in $\mathfrak{H}''$ so that, according to eq.(11A), the following is valid:

$$f(\xi) = \int\limits_{\Gamma} \Delta k_\xi(x) f(x)\,dx - \int\limits_{\Gamma} v(x)\, k_\xi(x)\, f(x)\,dx + \int\limits_{\Gamma} \left\{\frac{d}{dx} k_\xi \frac{d}{dx} f + v\, k_\xi f\right\} dx;$$

In addition, since k_ξ lies in $\mathfrak{G}$, eq.(6) will yield /701

$$f(\xi) = \int_{\Gamma} \Delta k_\xi(x) f(x)\, dx - \int_{\Gamma} v(x) k_\xi(x) f(x)\, dx + \int_{\Gamma} k_\xi h\, dx. \tag{12A}$$

The first two integrals, since v and f are continuous, are continuously differentiable to ξ; this is true also for the last integral according to lemma 4A.

Consequently, $f(\xi)$ is continuously differentiable to ξ and, according to lemma 2, f lies in $\mathfrak{G}''$.

Theorem 1.3A:

$\mathfrak{G}_1$ lies in $\mathfrak{F}''$.

Proof. Let f be an element from $\mathfrak{F}_1$ so that Gf lies in $\mathfrak{G}$. Then, according to theorem 1.1, Gf = h will lie in $\mathfrak{H}''$ and the representation (12A) is transformed into

$$f(\xi) = \tfrac{1}{2} \int_{\Gamma_\sigma} (x - \xi)\, \{h(x) - v(x) f(x)\}\, dx$$
$$+ \int_{\Gamma_{\sigma/2} - \Gamma_\sigma} \{k_\xi(x)\,(h(x) - v(x) f(x)) + \Delta k_\xi(x) f(x)\}\, dx. \tag{13A}$$

Since, now, h is also continuous, a double differentiation will yield the continuous function

$$\frac{d^2}{d\xi^2} f(\xi) = -h(\xi) + v(\xi) f(\xi)$$
$$+ \int_{\Gamma_{\sigma/2} - \Gamma_\sigma} \left\{\frac{d^2}{d\xi^2} k_\xi(x)\,(h(x) - v(x) f(x)) + \frac{d^2}{d\xi^2} \Delta k_\xi(x) f(x)\right\} dx.$$

According to lemma 3, f will then be located in $\mathfrak{F}''$.

8. Proof of Theorem 1B

B. Two and Three Independent Variables, n = 2; 3

Instead of the basic solutions with $|x - \xi| = \sqrt{\sum_{\nu=1}^{n} (x_\nu - \xi_\nu)^2}$

$$K(x - \xi) = -\frac{1}{2\pi} \ln |x - \xi| \qquad \text{for } n = 2,$$

$$K(x - \xi) = \frac{1}{4\pi} \frac{1}{|x - \xi|} \qquad \text{for } n = 3$$

let us consider the iterated basic solutions

$$K^2(x-\xi) = -\frac{1}{8\pi}|x-\xi|^2\{\ln|x-\xi|-1\} \qquad \text{for } n = 2,$$
$$K^2(x-\xi) = \frac{1}{8\pi}|x-\xi| \qquad \text{for } n = 3,$$

which are so selected that

$$\Delta K^2 = K$$

Then, we set /702

$$k_\xi = k_\xi(x) = s_\sigma(x)\, K^2(x-\xi).$$

Except at $x = \xi$, $k_\xi(x)$ is twice continuously differentiable; however, $\int\limits_\Gamma \Sigma \left(\frac{\partial}{\partial x} k_\xi(x)\right)^2 dx$ and $\int\limits_\Gamma (\Delta k_\xi(x))^2 dx$ also exist; $\Delta k_\xi(x)$, except at $x = \xi$, is twice continuously differentiable and $\Delta^2 k_\xi(x)$ is continuous. This is even more so the case for $\Delta^2 k_\xi(x) = 0$ in Γ_σ.

Consequently, k_ξ lies* in $\mathfrak{G}'$, Δk_ξ lies in $\mathfrak{H}'$, and $\Delta^2 k_\xi$ in $\mathfrak{H}''$. Let the function $v(x)\Delta k_\xi(x)$, lying in $\mathfrak{H}'$, be denoted by $v\Delta k_\xi$.

<u>Lemma 4B</u>. For all h from $\mathfrak{H}$, g from $\mathfrak{G}$,

$\int\limits_\Gamma \Delta^2 k_\xi h\, dx,$	is continuous
$\int\limits_\Gamma \Delta k_\xi h\, dx,$	is continuous
$\int\limits_\Gamma v\Delta k_\xi h\, dx,$	is continuous
$\int\limits_\Gamma k_\xi h\, dx,$	is twice continuously differentiable
$\int\limits_\Gamma \sum \frac{\partial}{\partial x} k_\xi \frac{\partial}{\partial x} g\, dx$	is continuously differentiable

in ξ from Γ_σ.

The arguments can again be considered as known for twice continuously differentiable functions h and g. Since these are dense in $\mathfrak{H}$ resp. $\mathfrak{G}$, with

* At $n = 2$ k_ξ lies in $\mathfrak{F}'$ and, at $n = 3$, in $\mathfrak{F}$.

H resp. G as measure, it is sufficient to demonstrate, at $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, that:

1) $$\Delta^2 k_{\xi+\epsilon}, \quad \Delta k_{\xi+\epsilon}, \quad v \Delta k_{\xi+\epsilon}, \quad \frac{\partial^2}{\partial \xi_\nu \partial \xi_\mu} k_{\xi+\epsilon}, \quad \frac{1}{|\epsilon|}\left(\frac{\partial}{\partial \xi_\nu} k_{\xi+\epsilon} - \frac{\partial}{\partial \xi_\nu} k_\xi\right)$$

are elements h_ε for which $\int_\Gamma h_\varepsilon^2 dx$ remains bounded in $|\epsilon|$;

2) $\frac{\partial}{\partial \xi_\nu} k_{\xi+\varepsilon}$, $\frac{1}{|\epsilon|}(k_{\xi+\varepsilon} - k_\xi)$ are elements g_ε, for which $\int_\Gamma \Sigma\left(\frac{\partial}{\partial x} g_\varepsilon\right)^2 dx$ remains bounded in $|\epsilon|$. This boundedness can be taken directly from the properties of k_ξ.

Again, we will form the mean-square values

$$\frac{1}{(2\delta)^n} \int_{|\eta - \xi| < \delta} k_\eta d\eta = k_{\xi,\delta}.$$

Then, we have the following: /703

Lemma 5B. The function $k_{\xi,\delta}(x)$ is three times continuously differentiable while $\Delta k_{\xi,\delta}$, except at $x = \xi$, is twice differentiable; we have

$$-\Delta k_{\xi,\delta}(x) = e_{\xi,\delta}(x) \quad \text{in } \Gamma_\sigma.$$

Hence, $\Delta k_{\xi,\delta}$ lies in $\mathfrak{F}'$.

Proof. If x is located external to Γ_σ, then $k_{\xi,\delta}$ is arbitrarily often continuously differentiable; however, within Γ_σ, we have

$$k_{\xi,\delta} = \int_{|\eta - \xi| < \delta} K^2(x - \eta)\, d\eta$$

and the required properties of this integral are known.

Lemma 6B. As $\delta \to 0$ and $\xi = \xi_\delta \to \xi_0$, for each h from $\mathfrak{H}$, we have

$$\int_\Gamma \Delta k_{\xi,\delta} h\, dx \to \int_\Gamma \Delta k_{\xi_0} h\, dx,$$

$$\int_\Gamma v \Delta k_{\xi,\delta} h\, dx \to \int_\Gamma v \Delta k_{\xi_0} h\, dx,$$

$$\int_\Gamma (\Delta^2 k_{\xi,\delta} + e_{\xi,\delta}) h\, dx \to \int_\Gamma \Delta^2 k_{\xi_0} h\, dx.$$

Proof. The first two statements follow from the fact that, uniformly in x,

$$\Delta k_{\xi,\delta}(x) \to \Delta k_{\xi}(x)$$

except in a vicinity $|x - \xi_0| < \epsilon$ and that there

$$\begin{aligned}
\int\limits_{|x-\xi_0|<\epsilon} (\Delta k_{\xi,\delta})^2 \, dx &= \int\limits_{|x-\xi_0|<\epsilon} \Big(\frac{1}{(2\delta)^n} \int\limits_{|\eta-\xi|<\delta} K(\eta - x)\, d\eta\Big)^2 dx \\
&\leqq \int\limits_{|x-\xi_0|<\epsilon} \frac{1}{(2\delta)^n} \int\limits_{|\eta-\xi|<\delta} (K(\eta-x))^2 \, d\eta \, dx \\
&\leqq \frac{1}{(2\delta)^n} \int\limits_{|x-\xi_0|<\epsilon} \int\limits_{|\eta-x|<\epsilon+2\delta+|\xi-\xi_0|} (K(\eta-x))^2 \, dx \, d\eta \\
&\leqq \int\limits_{|x'|<\epsilon+2\delta+|\xi-\xi_0|} (K(x'))^2 \, dx'
\end{aligned}$$

tends to zero simultaneously with ϵ and δ.

The third statement follows from the uniform convergence

$$\begin{aligned}
\Delta^2 k_{\xi,\delta} + e_{\xi,\delta} &= \Delta^2 k_{\xi,\delta} \to \Delta^2 k_{\xi_0} \text{ outside of } \Gamma_\sigma, \\
\Delta^2 k_{\xi,\delta} + e_{\xi,\delta} &= 0 \qquad = \Delta^2 k_{\xi_0} \text{ in } \Gamma_\sigma.
\end{aligned}$$

Now, we can directly prove the three theorems 1. /704

Theorem 1.1B:

$\mathfrak{F}_1$ lies in $\mathfrak{H}''$.

Proof. For f_1 from $\mathfrak{F}_1$, $Gf_1 = h$ from $\mathfrak{H}$, the following is valid, taking Section 5.2 into consideration and allowing for the fact that $\Delta k_{\xi,\delta}$ lies in $\dot{\mathfrak{F}}'$ and f_1 in $\mathfrak{G}$:

$$\begin{aligned}
\frac{1}{(2\delta)^n} \int\limits_{|x-\xi|<\delta} f_1 \, dx &= \int\limits_{\Gamma} (e_{\xi,\delta} + \Delta^2 k_{\xi,\delta}) f_1 \, dx + \int\limits_{\Gamma} \sum \frac{\partial}{\partial x} \Delta k_{\xi,\delta} \frac{\partial}{\partial x} f_1 \, dx \\
&= \int\limits_{\Gamma} (e_{\xi,\delta} + \Delta^2 k_{\xi,\delta}) f_1 \, dx - \int\limits_{\Gamma} v \Delta k_{\xi,\delta} f_1 \, dx + \int\limits_{\Gamma} \Delta k_{\xi,\delta} h \, dx,
\end{aligned}$$

since f_1 lies in $\mathfrak{F}_1$ and $\Delta k_{\xi,\delta}$ in $\mathfrak{F}'$ so that eq.(6) is applicable here.

Consequently, according to lemma 6B, the limiting value $f_{1\epsilon_0}$ exists for $\delta \to 0$ and $\xi = \xi_\delta \to \xi_0$, so that, in Γ_σ, we have

$$f_{1\xi} = \int_\Gamma \Delta^2 k_\xi f_1 \, dx - \int_\Gamma v \Delta k_\xi f_1 \, dx + \int_\Gamma \Delta k_\xi h \, dx. \tag{11B}$$

According to lemma 4B, $f_{1\xi}$ is continuous in Γ_σ and thus also in Γ; according to lemma 1, f_1 lies in $\mathfrak{H}''$ with $f_1(x) = f_{1x}$.

Theorem 1.2B:

$$\mathfrak{G}_1 \text{ lies in } \mathfrak{G}''.$$

Proof. Let g_1 lie in $\mathfrak{G}_1$, i.e., also in $\mathfrak{F}_1$, and let $Gg_1 = g$ lie in $\mathfrak{G}$.

According to theorem 1.1B, $g_1 = g_1(x)$ lies in $\mathfrak{H}''$ and, since $k_{\xi,\delta}$ lies in $\dot{\mathfrak{F}}'$, the following is valid according to Section 5.2:

$$g_1(\xi) = \int_\Gamma \{\Delta^2 k_\xi(x) - v(x) \Delta k_\xi(x)\} g_1(x) \, dx - \int_\Gamma \sum \frac{\partial}{\partial x} k_\xi \frac{\partial}{\partial x} g \, dx. \tag{12B}$$

The first integral is continuously differentiable in Γ_σ near ξ, which is true also for the last term in accordance with lemma 4B. Thus, $g_1(\xi)$ is continuously differentiable in Γ. According to lemma 2, $g_1(x)$ then lies in $\mathfrak{G}''$.

Theorem 1.3B:

$$\mathfrak{F}_2 \text{ lies in } \dot{\mathfrak{F}}'.$$

Proof. Let f_2 lie in $\mathfrak{F}_2$ and

$$G f_2 = f_1 \text{ in } \mathfrak{F}_1, \quad G^2 f_2 = h \text{ in } \mathfrak{H}.$$

According to theorems 1.1B and 1.2B, $f_1 = f_1(x)$ lies in $\mathfrak{H}''$ and $f_2 = f_2(x)$ in $\mathfrak{G}''$. The representation (12B) - because of k_ξ in $\mathfrak{G}$ and f_1 in $\mathfrak{F}_1$, eq.(6) applies here - will then assume the form /705

$$f_2(\xi) = \int_\Gamma \{\Delta^2 k_\xi(x) - v(x) \Delta k_\xi(x)\} f_2(x) \, dx + \int_\Gamma v(x) k_\xi(x) f_1(x) \, dx - \int_\Gamma k_\xi h \, dx. \tag{13B}$$

The twice continuous differentiability of the first two integrals follows in the known manner when using the continuous differentiability of $f_2(x)$; from lemma 4B, there follows the twice continuous differentiability of the last term. Consequently, according to lemma 3B, $f_2(x)$ lies in $\mathfrak{F}''$.

9. Discussion of the Spectrum

We will give simple conditions for the behavior of the auxiliary potential v in the case that the spectrum, in general or below a given bound, is an ordinary discrete spectrum. These conditions completely correspond to the Weyl's conditions (Bibl.12.2). In our method, we made use of a paper by Fr. Rellich (Bibl.8). Primarily, the theorems from Part I, Section 7 are used which are built up on the concept of complete continuity.

Theorem 5. For the case (3), (4) of the finite domain $\Gamma = K_R$: The spectrum of $G = -\Delta + v$ is discrete, and an infinitely increasing sequence of eigenvalues of finite multiplicity exists.

Theorem 6. For the case (1), (2) of the infinite domain: Let the auxiliary potential $v(x)$ increase uniformly to infinity as $r = |x| \to \infty$.

Then, the spectrum of $G = -\Delta + v$ will be discrete, and an infinitely increasing sequence of eigenvalues of finite multiplicity will exist.

Theorem 7*. For the case (1), (2) of the infinite domain: Let

$$\underline{\lim}\, v(x) = v_\infty \text{ for } |x| \to \infty.$$

Then, the spectrum of $G = -\Delta + v$ is discrete below v_∞.

As an essential aid, we will use Poincaré's inequality (Bibl.4.2) for domains $[x - \xi] < \delta$ from Γ [which, in the case (2), also are allowed to possess

* With respect to Schrödinger's hydrogen problem, see Courant-Hilbert (Bibl.2.7).

x = 0 as vertex]: A $c > 0$ exists so that, for all functions g from $\mathfrak{G}$, the following is valid:

$$\int_{[x-\xi]<\delta} g^2\,dx \leqq \frac{1}{(2\delta)^n}\Big\{\int_{[x-\xi]<\delta} g\,dx\Big\}^2 + \delta^2 c \int_{[x-\xi]<\delta} \sum \Big(\frac{\partial}{\partial x} g\Big)^2 dx. \qquad (14)$$

Proof. The proof, which is simple for domains $[x - \xi] < \delta$, is noted /706 here: It is sufficient to assume g as lying in $\mathfrak{G}'$. If $[x^1 - \xi] < \delta$, $[x^2 - \xi] < \delta$, we have

$$\begin{aligned}|g(x^1) - g(x^2)| \leqq &\int_{|x_1-\xi_1|<\delta} \Big|\frac{\partial}{\partial x_1} g(x_1, x_2^1, \ldots, x_n^1)\Big| dx_1 \\ &+ \int_{|x_2-\xi_2|<\delta} \Big|\frac{\partial}{\partial x_2}(x_1^2, x_2, x_3^1, \ldots, x_n^1)\Big| dx_2 \\ &+ \cdots \\ &+ \int_{|x_n-\xi_n|<\delta} \Big|\frac{\partial}{\partial x_n}(x_1^2, \ldots, x_{n-1}^2, x_n)\Big| dx_n.\end{aligned}$$

An integration to x^1 and x^2 will yield

$$\int_{[x^1-\xi]<\delta}\int_{[x^2-\xi]<\delta} |g(x^1) - g(x^2)|^2\,dx^1\,dx^2 \leqq (2\delta)^{n+2}.n \int_{[x-\xi]<\delta} \sum \Big(\frac{\partial}{\partial x} g\Big)^2 dx.$$

However, the left-hand side is nothing other than

$$2(2\delta)^n \int_{[x-\xi]<\delta} g^2\,dx - 2\Big\{\int_{[x-\xi]<\delta} g\,dx\Big\}^2.$$

This will produce eq.(14), with $c = 2n$.

This inequality will then lead to the following:

Lemma. The forms

$$\int_{K_\varrho} g^2\,dx, \quad \int_{K_\varrho} w(x)\,g^2\,dx \qquad\qquad \varrho < R$$

with the bounded function w(x), piecewise continuous in Γ and first explained in $\mathfrak{G}'$ and then continued on $\mathfrak{G}$, are completely continuous in $\mathfrak{G}$ with respect to G.

<u>Proof</u>. The domain K_ρ can be covered by a finite sum $\sum_\xi$ of strange squares $[x - \xi] < \delta$ with arbitrarily small δ. By summation of the Poincaré inequality, we obtain

$$\int_{K_\varrho} g^2 dx \leqq (2\delta)^n \sum_\xi \left\{ \int_{[x-\xi]<\delta} g\, dx \right\}^2 + 2\delta^2 n \int_\Gamma \sum \left(\frac{\partial}{\partial x} g\right)^2 dx.$$

Taking eq.$(3)_\Theta$ according to the definition of Part I into consideration, this means that $\int_{K_\rho} g^2 dx$ is completely continuous with respect to **G**. The complete continuity of $\int_{K_\rho} wg^2 dx$ is then directly obtained from the fact that, with $|w(x)| \leq \overline{w}$,

$$\left| \int_{K_\varrho} w g^2 dx \right| \leqq \overline{w} \int_{K_\varrho} g^2 dx$$

can be estimated, in which case the right-hand side is recognized as completely continuous.

For proving theorem 5, we refer to the lemma 18 in Part I. Thus, it is /70/ sufficient to prove the complete continuity of $\int_\Gamma g^2 dx$ with respect to **G**. We resolve

$$\int_\Gamma g^2 dx = \int_{K_{R-\sigma}} g^2 dx + \int_{K_{R-\sigma,\, R}} g^2 dx$$

and make use of the inequality (1.1) in the Appendix, with $u = g$.

After averaging there over ρ from $R - \sigma$ to R and over τ from $R - \sigma$ to $R - \frac{\sigma}{2}$, in which case it is assumed that $R - \sigma > \frac{R}{2}$, the following is obtained from logical estimates:

$$\int_{K_{R-\sigma,\, R}} g^2 dx \leqq c_1 \int_{K_{R-\sigma/2,\, R-\sigma}} g^2 dx + \sigma^2 c_2 \int_{K_R} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx$$

for example, with $c_1 = 2^{n+1}$, $c_2 = 2R^{n-1} C$.

By substitution, we obtain

$$\int_\Gamma g^2 dx \leqq \int_{K_{R-\sigma}} g^2 dx + c_1 \int_{K_{R-\sigma/2,\, R-\sigma}} g^2 dx + \sigma^2 c_2 \int_{K_R} \sum \left(\frac{\partial}{\partial x}\right)^2 dx$$

and thus also the statement, since the first two terms on the right-hand side are completely continuous according to the auxiliary theorem.

For proving theorem 6, it is sufficient according to lemma 18 in Part I, to demonstrate that $\int_\Gamma g^2 dx$ is G completely continuous. Let

$$m(\varrho) = \operatorname{Min} v(x)$$

for $x \geq \rho$; then, the postulate of theorem 6 means that

$$m(\varrho) \to \infty \text{ for } \varrho \to \infty.$$

Now, the following is obviously valid:

$$\int_\Gamma g^2\, dx \leqq \int_{K_\varrho} g^2\, dx + \frac{1}{m(\varrho)} \int_{\Gamma - K_\varrho} v g^2\, dx$$

and, in addition (see footnote on p.36),

$$\int_{\Gamma - K_\varrho} v g^2\, dx \leqq g G g.$$

This constitutes the statement.

For proving theorem 7, we resolve v(x) into

$$v(x) = v^+(x) - w(x),$$

where $v^+(x)$ and w(x) are continuous functions in Γ, selected so that

$$v^+(x) \geqq v_\infty, \quad 0 \leqq w(x) \leqq v_\infty$$

in such a manner that we have /708

$$w(x) \to 0 \quad \text{for} \quad |x| \to \infty.$$

In the case $(2)_{\Theta}$, this argument is to exist only for $|x| \geq P$; conversely, for $0 < |x| \leq P$, we stipulate that

$$v^+(x) = v(x) + v_\infty, \quad w(x) = v_\infty,$$

which is possible by suitable selection without violating the continuity at

$|x| = P$.

According to these findings, the form

$$g W g = \int_{\Gamma} w(x) g^2 \, dx$$

in $\mathfrak{G}$ is bounded so that the following applies to the form defined in $\mathfrak{G}$:

$$g G^+ g = \int_{\Gamma} \left\{ \sum \left(\frac{\partial}{\partial x} g \right)^2 + v^+ g^2 \right\} dx$$
$$g G^+ g \geqq v_\infty \int_{\Gamma} g^2 \, dx, \tag{$*$}$$

where, in the case $(2)_{\Theta}$, reference has been made to the estimate (4). Since obviously

$$v_\infty (g G g) \geqq g G^+ g \geqq g G g \tag{$\overset{*}{*}$}$$

is valid, G^+ in $\mathfrak{G}$ is closed.

Since v^+ accurately satisfies the conditions of an auxiliary potential, the spectral analysis is applicable to the form G^+. Since its lower bound evidently is v_∞, this form will have no spectrum below v_∞. Theorem 7, namely, that the spectrum of the operator G or of the form $G = G^+ + W$ is discrete below v_∞ then follows directly from theorem 17 (I), provided that it can be demonstrated that W is completely continuous with respect to G^+ in $\mathfrak{G}$ or - which is equivalent because of eq.($\overset{*}{*}$) - with respect to G.

If then

$$m(\varrho) = \operatorname{Max} w(x) \text{ for } |x| \geqq \varrho,$$

it follows from $w(x) \to 0$ for $|x| \to \infty$ that

$$m(\varrho) \to 0, \quad \text{for} \quad \varrho \to \infty.$$

We now have

$$0 \leqq \int_{\Gamma} w g^2 \, dx \leqq v_\infty \int_{K_\varrho} g^2 \, dx + m(\varrho) \int_{\Gamma - K_\varrho} g^2 \, dx.$$

Because of $\int_{\Gamma} g^2 dx \leq gGg$, this constitutes the complete continuity of W according to the definition given in Part I.

10. Appendix

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1. Integral Inequalities

Let u(x) be a function of $x = (x_1, \ldots x_n)$ piecewise continuously differentiable in $0 < r < R$. Then, the inequality

$$\left| \sqrt{\int\limits_{\Omega_\varrho} u^2 d\omega} - \sqrt{\int\limits_{\Omega_\tau} u^2 d\omega} \right| \leqq \sqrt{C|\varrho - \tau|} \sqrt{\int\limits_{K_{\tau,\varrho}} \sum \left(\frac{\partial u}{\partial x}\right)^2 dx} \tag{1.0}$$

exists for

$$C \geqq \frac{1}{\varrho - \tau} \int\limits_{\tau}^{\varrho} \frac{dr}{r^{n-1}}$$

with $0 < \rho < P$, $0 < \tau < P$, $\tau < \rho$.

Proof. Let $x = \xi$ with $|\xi| = 1$ be a point on Ω_1; then,

$$u(\varrho\,\xi) - u(\tau\,\xi) = \int\limits_{\tau}^{\varrho} \frac{\partial}{\partial r} u(r\,\xi)\,dr,$$

$$u^2(\varrho\,\xi) - 2\,u(\varrho\,\xi)\,u(\tau\,\xi) + u^2(\tau\,\xi) \leqq \int\limits_{\tau}^{\varrho} \frac{dr}{r^{n-1}} \int\limits_{\tau}^{\varrho} r^{n-1} \left(\frac{\partial}{\partial r} u(r\,\xi)\right)^2 dr.$$

By integrating over Ω_1 and applying the Schwarz inequality to the left-hand side, we obtain

$$\left| \sqrt{\int\limits_{\Omega_\varrho} u^2 d\omega} - \sqrt{\int\limits_{\Omega_\tau} u^2 d\omega} \right|^2 \leqq \int\limits_{\tau}^{\varrho} \frac{dr}{r^{n-1}} \int\limits_{K_{\tau,\varrho}} \sum \left(\frac{\partial u}{\partial x}\right)^2 dx$$

and, from this, the inequality (1.0).

The inequality (1.0) will then yield

$$\int\limits_{\Omega_\varrho} u^2 d\omega \leqq 2 \int\limits_{\Omega_\tau} u^2 d\omega + 2C|\varrho - \tau| \int\limits_{K_{\tau,\varrho}} \sum \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{1.1}$$

with $K_{\rho,\tau}$ instead of $K_{\tau,\rho}$ for $\tau > \rho$.

2. Singularity of the Auxiliary Potential

Let u(x) be piecewise continuously differentiable for $0 < r \leq P$

$$x = x_1, \ldots, x_n; \quad n = 1, 2, 3 \ldots.$$

Then, the following inequality exists for $0 < \sigma < \rho \leq P$: /710

$$T_{\sigma,\varrho} \equiv \int_{K_{\sigma,\varrho}} \left\{ \sum \left(\frac{\partial u}{\partial x}\right)^2 - \varphi^2(r) u^2 \right\} dx - \varphi(\sigma)\sigma^{n-1} \int_{\Omega_\sigma} u^2 d\omega \tag{2.0}$$
$$\geqq -\varphi(\varrho)\varrho^{n-1} \int_{\Omega_\varrho} u^2 d\omega$$

with

$$\varphi(r) = \frac{n-2}{2}\frac{1}{r} \qquad \text{for } n = 3, 4, \ldots$$
$$\varphi(r) = \frac{1}{2}\frac{1}{\ln\frac{A}{r}}\frac{1}{r} \qquad \text{for } n = 2 \quad \text{with } A > P.$$

Proof. We have

$$T_{\sigma,\varrho} = \int_{K_{\sigma,\varrho}} \frac{1}{\varphi(r) r^{n-1}} \sum \left[\frac{\partial}{\partial x}\left(r^{\frac{n-1}{2}} \sqrt{\varphi(r)}\, u\right)\right]^2 dx - \varphi(\varrho)\varrho^{n-1} \int_{\Omega_\varrho} u^2 d\omega .$$

Lemmas:

(2.1). The quantity

$$T_{\sigma,\varrho}$$

does not decrease as $\sigma \to 0$.

The proof follows directly from the above presentation of $T_{\sigma,\rho}$.

(2.2). One positive constant k exists so that, for sufficiently small $\sigma (\sigma \leq \sigma_0)$, the following holds:

$$T_{\sigma,P} \geqq -k \int_{K_{\sigma,P}} u^2 dx.$$

Proof. Let σ_1 be such a number between σ_0 and P that we obtain

$$\sigma_1^{n-1} \int_{\Omega_{\sigma_1}} u^2 d\omega = \frac{1}{P - \sigma_0} \int_{K_{\sigma_0,P}} u^2 dx$$

Then, for $\sigma \leq \sigma_0$ and because of $T_{\sigma,P} \geq T_{\sigma_1,P}$, we obtain

$$T_{\sigma,P} \geqq -\frac{\varphi(\sigma_1)}{P-\sigma_0} \int\limits_{K_{\sigma_0,P}} u^2\,dx - \int\limits_{K_{\sigma_1,P}} \varphi^2(r)\,u^2\,dx.$$

If, now, $k = \text{Max}\left[\frac{\varphi(\sigma_1)}{P-\sigma_0} + \varphi^2(r)\right]$ in $\sigma_0 \leqq r \leqq P$, it follows that

$$T_{\sigma,P} \geqq -k \int\limits_{K_{\sigma_0,P}} u^2\,dx$$

and thus also the statement.

(2.3). If $\int\limits_{K_\rho} \Sigma \left(\frac{\partial u}{\partial x}\right)^2 dx$ exists, then for $n \geq 2$,

$$\varphi(\sigma)\,\sigma^{n-1} \int\limits_{\Omega_\sigma} u^2\,d\omega \underset{\sigma\to 0}{\to} 0$$

and /711

$$\int\limits_{K_\rho} \varphi^2(r)\,u^2\,dx.$$

exists.

Proof. From eq.(2.0), because of $\varphi(r) > 0$, the boundedness of $\int\limits_{K_{\sigma,\rho}} \varphi^2(r)u^2dx$ in σ follows directly and thus also the existence of

$$\int\limits_{K_\rho} \varphi^2(r)\,u^2\,dx = \int\limits_0^\rho \varphi(r)\left[\varphi(r)\,r^{n-1} \int\limits_{\Omega_r} u^2\,d\omega\right] dr.$$

From the existence of this integral and the nonexistence of

$$\int\limits_0^\rho \varphi(r)\,dr$$

it follows that a special sequence $\rho \to 0$ exists for which

$$\varphi(\varrho)\,\varrho^{n-1} \int\limits_{\Omega_\varrho} u^2\,d\omega \underset{\varrho\to 0}{\to} 0$$

From eq.(2.0) we can then conclude that, for each sequence, $\sigma \to 0$ applies.

3) Proof of Green's Transformation

To prove

$$\int\limits_\Gamma \left\{\sum \frac{\partial}{\partial x} f \frac{\partial}{\partial x} g + v f g\right\} dx - \int\limits_\Gamma \{\Delta f g + v f g\}\,dx = 0 \tag{6}$$

for f in $\mathfrak{F}$ and g in $\mathfrak{G}'$, we first will integrate only over $K_{\sigma,\rho}$:

$$\int_{K_{\sigma,\varrho}} \left\{ \sum \frac{\partial}{\partial x} f \frac{\partial}{\partial x} g + v f g + \Delta f g - v f g \right\} dx$$
$$= \varrho^{n-1} \int_{\Omega_\varrho} g \frac{\partial}{\partial r} f \, d\omega - \sigma^{n-1} \int_{\Omega_\sigma} g \frac{\partial}{\partial r} f \, d\omega,$$

where $\sigma \neq 0$ need be selected only in the case (2). Then, it must be demonstrated that special sequences $\rho \to \infty$ resp. $\rho \to R$ and $\sigma \to 0$ exist for which the right-hand sides vanish.

Case (1): From the existence of the integral

$$\int_0^\infty r^{n-1} \int_{\Omega_r} \left| g \frac{\partial}{\partial r} f \right| d\omega \, dr \leq \sqrt{\int_\Gamma \sum \left(\frac{\partial}{\partial x} f \right)^2 dx \int_\Gamma g^2 \, dx}$$

there follows the existence of a sequence $\rho \to \infty$, for which

$$\varrho^n \int_{\Omega_\rho} g \frac{\partial f}{\partial r} d\omega \to 0$$

Case (2): The sequence $\rho \to \infty$ is selected as in the case (1). /712

From the existence of $\int_{K_P} \varphi^2(r) g^2 dx$ and from the existence of

$$\int_{K_P} \sum \left(\frac{\partial}{\partial x} f \right)^2 dx$$

the existence of

$$\int_{K_P} \varphi(r) \left| g \frac{\partial f}{\partial r} \right| dx = \int_0^P \varphi(r) \left[r^{n-1} \int_{\Omega_r} \left| g \frac{\partial f}{\partial r} \right| d\omega \right] dr$$

is obtained. From this existence and from the nonexistence of $\int_0^P \varphi(r)dr$, it can be concluded that a sequence $\sigma \to 0$ exists for which

$$\sigma^{n-1} \int_{\Omega_\sigma} \left| g \frac{\partial f}{\partial r} \right| d\omega \to 0$$

as desired.

Case (3): See also Courant (Bibl.2.5); from the existence of the integral

$$\int_0^R r^{n-1} \int_\Omega \sum \left(\frac{\partial}{\partial x} f\right)^2 d\omega\, dr$$

there follows the existence of a sequence $\rho \to R$, such that

$$(R-\varrho) \int_{\Omega_\varrho} \sum \left(\frac{\partial}{\partial x} f\right)^2 d\omega \to 0.$$

From the boundary condition (1*) on p.46,

$$\int_{\Omega_\varrho} g^2 d\omega \leqq 2C(R-\varrho) \int_{K_{\varrho,R}} \sum \left(\frac{\partial}{\partial x} g\right)^2 dx \qquad (1^*)$$

it follows on the other hand that $\frac{1}{R-\rho} \int_{\Omega_\rho} g^2 d\omega \to 0$.

Consequently,

$$\int_{\Omega_\varrho} \left| g \frac{\partial f}{\partial r} \right| d\omega \leqq \sqrt{\int_{\Omega_\varrho} \sum \left(\frac{\partial f}{\partial x}\right)^2 d\omega \int_{\Omega_\varrho} g^2 d\omega} \underset{\varrho \to R}{\to} 0.$$

Case (4): Here, the boundary condition already means the required convergence

$$\varrho^{n-1} \int_{\Omega_\varrho} g \frac{\partial f}{\partial r} d\omega \to 0. \qquad (5)$$

Auxiliary theorem. In the case (1), it follows, for piecewise twice /713 continuously differentiable functions f(x) from $\mathfrak{H}''$ for which $\Delta f(x) + v(x)f(x)$ is a function from $\mathfrak{H}'$, that

$$\int_\Gamma \left\{ \sum \left(\frac{\partial}{\partial x} f(x)\right)^2 + v(x) f^2(x) \right\} dx$$

exists, so that f lies in $\mathfrak{F}'$.

This is so since

$$\int_{K_\varrho} \left\{ \sum \left(\frac{\partial}{\partial x} f(x)\right)^2 + v(x) f^2(x) \right\} dx$$

$$= \int_{K_\varrho} f(x)\left(-\Delta f(x) + v(x) f(x)\right) dx + \varrho^{n-1} \int_{\Omega_\varrho} f \frac{\partial}{\partial r} f\, d\omega.$$

Because of the existence of $\int_0^\infty r^{n-1} \int_{\Omega_r} f^2 d\omega dr$, a sequence $\rho \to \infty$ must exist in whose vicinity $\int_{\Omega_r} f^2 d\omega$ decreases so that there

$$\frac{d}{dr}\int_{\Omega_r} f^2\, d\omega = 2\int_{\Omega_r} f\,\frac{\partial}{\partial r} f\, d\omega$$

becomes negative. Consequently, for this sequence, the left-hand side remains bounded and, since the integrand is positive, the integral exists over $K_\infty = \Gamma$.

It is demonstrated here that the case (1) represents the limiting point case in the sense of Weyl.

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